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Excursion Set Theory for generic moving barriers and non-Gaussian initial conditions

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ABSTRACT

Excursion set theory, where density perturbations evolve stochastically with the smoothing scale, provides a method for computing the mass function of cosmological structures like dark matter halos, sheets and filaments. The computation of these mass functions is mapped into the so-called first-passage time problem in the presence of a moving barrier. In this paper we use the path integral formulation of the excursion set theory developed recently to analytically solve the first-passage time problem in the presence of a generic moving barrier, in particular the barrier corresponding to ellipsoidal collapse. We perform the computation for both Gaussian and non-Gaussian initial conditions and for a window function which is a top-hat in wavenumber space. The expression of the halo mass function for the ellipsoidal collapse barrier and with non-Gaussianity is therefore obtained in a fully consistent way and it does not require the introduction of any form factor artificially derived from the Press-Schechter formalism based on the spherical collapse and usually adopted in the literature.

Key words: cosmology: theory – large scale structure of the universe

1 INTRODUCTION

The mass function of dark matter halos is a central object in modern cosmology, because of its relevance to the formation and evolution of galaxies and clusters. It is therefore important to have accurate theoretical predictions for it, first of all when the primordial fluctuations are taken to be Gaussian, and then when some level of non-Gaussianity is included. Non-Gaussianities are particularly relevant in the high-mass end of the power spectrum of perturbations, i.e. on the scale of galaxy clusters, since the effect of non-Gaussian (NG) fluctuations becomes especially visible on the tail of the probability distribution. As a result, both the abundance and the clustering properties of very massive halos are sensitive probes of primordial non-Gaussianities (Matarrese et al. 1986; Grinstein & Wise 1986; Lucchin et al. 1988; Moscardini et al. 1991; Koyama et al. 1999; Matarrese et al. 2000; Robinson & Baker 2000; Robinson et al. 2000; LoVerde et al. 2008; Maggiore & Riotto 2010c; Lam & Sheth 2009; Giannantonio & Porciani 2010), and could be detected or significantly constrained by the various planned large-scale galaxy surveys, both ground based (such as DES, PanSTARRS and LSST) and in space (such as EUCLID and ADEPT) see, e.g. Dalal et al. (2008) and Carbone et al. (2008). Furthermore, the primordial NG alters the clustering of dark matter halos inducing a scale-dependent bias on large scales (Dalal et al. 2008; Matarrese & Verde 2008; Slosar et al. 2008; Afshordi & Tolley 2008) while even for small primordial NG the evolution of perturbations on super-Hubble scales yields extra contributions on smaller scales (Bartolo et al. 2005; Matarrese & Verde 2009).

The formation and evolution of dark matter halos is a highly complex phenomenon, and a detailed quantitative understanding of it can only come through large-scale N-body simulations, such as the Millennium simulation (Springel et al. 2005). Simulations with non-Gaussian initial conditions have also been performed (Grossi et al. 2009; Giannantonio & Porciani 2010; Wagner et al. 2010). At the same time, some analytic understanding of the process of halo formation is also desirable, both for the deeper physical understanding that analytic models offer, and for their flexibility under changes of parameters of the cosmological model, shape of non-Gaussianities, etc. Analytical derivations of the halo mass function are typically based on Press-Schechter (PS) theory (Press & Schechter 1974) and its extension (Peacock & Heavens 1990; Bond et al. 1991) known as excursion set theory (see Zentner (2007) for a recent review). In excursion set theory the density perturbation evolves stochastically with the smoothing scale, and the problem of computing the

probability of halo formation is mapped into the so-called first-passage time problem in the presence of a barrier.

The original formulation of excursion set theory (Bond et al. 1991) makes a number of simplifying assumptions, both at the technical level, and concerning the physics of halo formation. In particular, at the technical level it is assumed that the smoothed density field δ evolves with the smoothing scale R (or more precisely with the variance $S(R)$ of the smoothed density field) in a Markovian way. However, this assumption is correct only if the density field is smoothed with a window function which is a top-hat in wavenumber space, and with such a smoothing function it is difficult to associate a mass M to a region smoothed with smoothing parameter R, so in practice it is not possible to associate a mass to the dark matter halos identified in this way. For any other choice of the window function (such as a top-hat in real space, for which the relation between the mass M and the smoothing scale R is trivially $M = (4/3)\pi R^3 \bar{\rho}$, where $\bar{\rho}$ is the average density of the universe), the actual evolution of the smoothed density field with R is non-Markovian. At the physical level, the crucial simplifying assumption of the original formulation of excursion set theory is that dark matter halo forms through the spherical collapse of initial overdensities. However the actual process of halo formation, as revealed by N-body simulations, is much more complicated, and involves smooth accretion, tidal interactions with the environment, as well as violent episodes of collisions with other halos, merging and fragmentation.

In a recent series of papers (Maggiore & Riotto 2010a,b,c) (hereafter MR1, MR2 and MR3, respectively), the original formulation of excursion set theory has been extended to deal with the non-Markovian effects which are induced either by the use of a realistic filter function, or by non-Gaussianities in the primordial density field. The basic idea is to reformulate the first-passage time problem in the presence of a barrier in terms of the computation of a path integral with a boundary (i.e. over a sum over all "trajectories" $\delta(S)$ that always stay below the barrier), and then to use standard results from quantum field theory and statistical mechanics to express this path integral in terms of the connected correlators of the theory. A path-integral with boundaries of the kind that we obtain is however not a very common object even in quantum field theory or statistical mechanics, and in MR1 and MR3 we developed the technique for evaluating it perturbatively with respect to the non-Markovian and the non-Gaussian effects. This provided first of all a rederivation of the results of excursion set theory which, from the mathematical point of view, is from first principles (for instance the absorbing barrier boundary condition, which in the original formulation was imposed by hand, comes out automatically in the formalism of MR1). Furthermore it allows us to include, at least perturbatively, the effect of non-Markovianities and of non-Gaussianities. In particular, in MR3 we have shown how to include the effect of a non-vanishing bispectrum, while the case of a non-vanishing trispectrum was considered in Maggiore & Riotto (2010d) (see also D'Amico et al. (2010) for an approach to non-Gaussianities which combines our technique with the saddle point method developed in Matarrese et al. (2000)).

Of course this extension of excursion set theory, even if it provides an improvement of the original formulation from the mathematical point of view, still shares the same physical limitations of the original formulations, as long as the same model for collapse is used. The model for collapse can be improved in different, complementary, ways. A crucial step was taken by Sheth, Mo & Tormen (2001) who took into account the fact that actual halos are triaxial (Bardeen et al. 1986; Bond & Myers 1996) and showed that an ellipsoidal collapse model can be implemented, within the excursion set theory framework, by computing the first-crossing rate in the presence of a barrier $B_{\rm el}(S)$ which depends on S ("moving barrier"), rather than being constant at the value δ_c of the spherical collapse,

$$B_{\rm el}(S) \simeq \delta_c \left[1 + 0.4 \left(\frac{S}{\delta_c^2} \right)^{0.6} \right] . \tag{1}$$

Physically this reflects the fact that low-mass halos (which corresponds to large S) have larger deviations from sphericity and significant shear, that opposes collapse. Therefore low-mass halos require a higher density to collapse. In contrast, very large halos are more and more spherical, so their effective barrier reduces to the one for spherical collapse. In order to improve the agreement between the prediction from the excursion set theory with an ellipsoidal collapse and the N-body simulations, Sheth, Mo & Tormen (2001) also found that it was necessary to replace δ_c with $\sqrt{a}\delta_c$, where $\sqrt{a} \simeq 0.84$ was obtained by requiring that their mass function fits the GIF simulation. The moving barrier therefore becomes

$$B_{\rm el}(S) \simeq \sqrt{a}\ \delta_c \left[1 + 0.4 \left(\frac{S}{a\, \delta_c^2} \right)^{0.6} \right] . \tag{2}$$

The parameter a cannot be derived from the dynamics of the ellipsoidal collapse. Rather on the contrary, the ellipsoidal collapse model would predict $a = 1$ because in the limit $S \equiv \sigma^2 \to 0$ (i.e. in the large mass limit) halos become more and more spherical, and therefore the barrier must reduce to that of spherical collapse. This mismatch might be originated by the fact that, as mentioned above, halo collapse is a very complex dynamical phenomenon, and modeling it as spherical, or even as ellipsoidal, is a significant oversimplification. In addition, the very definition of what is a dark matter halo, both in N-body simulations and observationally, is a difficult problem. In MR2 it was proposed that some of the physical complications inherent to a realistic description of halo formation can be included in the excursion set theory framework, at least at an effective level, by taking into account that the critical value for collapse is itself a stochastic variable, whose scatter reflects a number of complicated aspects of the underlying dynamics (see also Audit et. al. (1997); Lee & Shandarin (1998); Sheth, Mo & Tormen (2001) for earlier related ideas). Solving the first-passage time problem in the presence of a barrier which is diffusing around the value δ_c of the spherical collapse model, it was found in MR2 that the exponential factor in the Press-Schechter mass function changes from $\exp\{-\delta_c^2/2\sigma^2\}$ to $\exp\{-a\delta_c^2/2\sigma^2\}$, where $a = 1/(1 + D_B)$ and D_B is the diffusion coefficient of the barrier. The numerical value of D_B, and therefore the corresponding value of a, depends among other things on the algorithm used for identifying halos. From recent N-body simulations that studied the properties of the collapse barrier, a value $D_B \simeq 0.25$ was deduced in MR2 predicting $a \simeq 0.80$ (up to σ smaller than

about 3) We remark that the deduced value of a also holds when the collapse is ellipsoidal which was a good fit to the average threshold barrier found by N-body data. The value of $a \simeq 0.80$ isin excellent agreement with the exponential fall off of the mass function found in N-body simulations, for the same halo definition.

The path-integral formulation developed in MR1 and MR3 was restricted to the case of a constant barrier δ_c (while in MR2 were considered the stochastic fluctuations around it). The aim of this paper is to extend the path integral formulation of excursion set theory to the case of a generic moving barrier, and to provide analytical expressions which can be used to calculate the corresponding first-passage time probability.

Given that the Sheth-Tormen (ST) halo mass function is widely used in the literature, we believe that it is interesting to derive it by computing the first-crossing rate with an ellipsoidal barrier from first principles. To the best of our knowledge, an analytical expression of the first-crossing rate was given in Sheth & Tormen (2002) just as a fit to the N-body data and its derivation has been sketched only recently in Lam & Sheth (2009). As we shall see, this derivation is not free from drawbacks. There are other good reasons why solving analytically for the first-crossing rate with a generic moving barrier is interesting. First, excursion set theory can be applied to characterize the cosmic web (Shen et al. (2006)). Combining models of triaxial collapse with excursion set theory, cosmic sheets are defined as objects that have collapsed along only one axis, filaments have collapsed along two axes, and halos are objects in which triaxial collapse is complete. Computing the abundances of cosmic sheets, filaments and halos within the excursion set theory amounts again to solving a first-time passage problem with the corresponding moving barriers

$$B_{\rm sheet}(S) \simeq \sqrt{a}\;\delta_c \left[1 - 0.56 \left(\frac{S}{a\,\delta_c^2}\right)^{0.55}\right], \tag{3}$$

$$B_{\rm filam}(S) \simeq \sqrt{a}\;\delta_c \left[1 - 0.012 \left(\frac{S}{a\,\delta_c^2}\right)^{0.28}\right]. \tag{4}$$

The insertion of each moving barrier into the excursion set approach provides estimates of the mass fraction in sheets, filaments and halos as a function of mass and time. Secondly, moving barriers are adopted in modelling through the excursion set method the sizes of ionized regions during the epoch of reonization (Furlanetto et al. (2004)), while Sheth & Tormen (2002) suggested that moving barriers could effectively incapsulate a wide variety of phenomena such as suppression of the collapse of small, low-mass, overdense patches in models in which dark matter is warm. For a given choice of the barrier, the first-crossing rate can in principle be evaluated with numerical techniques (Bond et al. 1991; Zhang & Hui 2006), but it interesting to obtain analytic formulas valid for a generic functions $B(S)$. Thirdly, as we already mentioned, it has become recently clear that detecting a significant amount of non-Gaussianity and its shape either from the Cosmic Microwave Background (CMB) or from the Large Scale Structure (LSS) offers the possibility of opening a window into the dynamics of the universe during the very first stages of its evolution (Bartolo et al. (2004)). It is therefore of primary importance to compute the halo mass function when NG initial conditions are present. The halo mass function with NG has been calculated in Matarrese et al. (2000) and LoVerde et al. (2008) using the PS approach with a spherical collapse, while the path integral formulation of excursion set theory in the presence of NG and with a diffusive barrier has been formulated in MR3. The main motivation to compute the halo mass function in the presence of NG within the excursion set method and with a moving ellipsoidal barrier is dictated by the fact that it has become customary in the literature to obtain the halo mass function with NG by multiplying the ST halo mass function with gaussian initial conditions by a form factor obtained by dividing the first-crossing rate with NG obtained for the PS spherical collapse case (Matarrese et al. (2000); LoVerde et al. (2008)) by the PS one (the exception is represented by the consistent calculation of MR3, which does not require this procedure). It is unclear (at least to us) why and to which extent this spurious method should provide a good approximation to the correct halo mass function with NG and ellipsoidal barrier. The issue is also timely since N-body data with NG initial conditions finally exist (Grossi et al. 2009; Giannantonio & Porciani 2010; Wagner et al. 2010), and may be compared to the various theoretical predictions for the halo mass functions with NG. They differ at the $\mathcal{O}(20)\%$ level and it is important to understand which error is introduced by adopting the form factor procedure.

The paper is organized as follows. In section 2 we review the approach to the computation of the halo mass function based on excursion set theory. In particular, in section 2.1 we begin with a quick review of the case in which the collapse is assumed to be spherical, primordial fluctuations are taken to be Gaussian, and the evolution of the density perturbation with the smoothing scale is assumed to be Markovian. This is the setting considered in the classical paper by Bond et al. (1991). We will then proceed toward increasing complexity. In Section 2.2 we review the the basic points of the approach developed in MR1, MR2 and MR3. In Section 3 we present the computation of the first crossing rate for a generic moving barrier, while Section 4 contains the generalization of the computation to the case of NG initial conditions. Various technical details are collected in Appendices A-D.

2 THE HALO MASS FUNCTION IN EXCURSION SET THEORY

The halo mass function can be written as

$$\frac{dn(M)}{dM} = f(\sigma)\frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}(M)}{d\ln M}\,, \tag{5}$$

where $n(M)$ is the number density of dark matter halos of mass M, $\sigma(M)$ is the variance of the linear density field smoothed on a scale R corresponding to a mass M, and $\bar{\rho}$ is the average density of the universe. The basic problem is therefore the computation of the function $f(\sigma)$.

2.1 Spherical collapse, Gaussian fluctuations, and Markovian evolution with the smoothing scale

Let us summarize the basic points of the original formulation of excursion set theory. One considers the density field

δ smoothed over a radius R, and studies its stochastic evolution as a function of the smoothing scale R. As it was found in the classical paper by Bond et al. (1991), when the density $\delta(R)$ is smoothed with a sharp filter in momentum space, and the density fluctuations have Gaussian statistics, the smoothed density field satisfies the equation

$$\frac{\partial \delta(S)}{\partial S} = \eta(S) , \tag{6}$$

where $S = \sigma^2(R)$ is the variance of the linear density field smoothed on the scale R and computed with a sharp filter in momentum space, while $\eta(S)$ is a stochastic variable that satisfies

$$\langle \eta(S_1)\eta(S_2)\rangle = \delta_D(S_1 - S_2) , \tag{7}$$

where δ_D denotes the Dirac delta function. Equations (6) and (7) are the same as a Langevin equation with a Dirac-delta noise $\eta(S)$, with the variance S formally playing the role of time. Let us denote by $\Pi(\delta, S)d\delta$ the probability density that the variable $\delta(S)$ reaches a value between δ and $\delta + d\delta$ by "time" S. A textbook result in statistical physics is that, if a variable $\delta(S)$ satisfies a Langevin equation with a Dirac-delta noise, the probability density $\Pi(\delta, S)$ satisfies the Fokker-Planck (FP) equation

$$\frac{\partial \Pi}{\partial S} = \frac{1}{2}\frac{\partial^2 \Pi}{\partial \delta^2} . \tag{8}$$

The solution of this equation over the whole real axis $-\infty < \delta < \infty$, with the boundary condition that it vanishes at $\delta = \pm\infty$, is

$$\Pi^0(\delta, S) = \frac{1}{\sqrt{2\pi S}}\, e^{-\delta^2/(2S)} . \tag{9}$$

and is nothing but the distribution function of PS theory. Since, in hierarchical models of structure formation, as R increases, i.e. as the halo mass increases, the variance S decreases monotonically, in Bond et al. (1991) it was realized that we are actually interested in the stochastic evolution of δ against S only until the "trajectory" crosses for the first time the threshold δ_c for collapse. The threshold value δ_c is estimated within the spherical collapse model where a spherically symmetric inhomogeneity behaves like a closed collapsing universe. The underlying idea behind the PS theory is that the comoving number density of collapsed haloes can computed from the statistical properties of the linear density field, assumed to be Gaussian. In this picture haloes form when the smoothed linear density contrast is larger than $\delta_c \simeq 1.68$ which is obtained computing the linear density contrast at the collapse time. This result can be extended to arbitrary redshift z by reabsorbing the evolution of the variance into δ_c, so that δ_c in the above result is replaced by $\delta_c(z) = \delta_c(0)/D(z)$, where $D(z)$ is the linear growth factor. Notice that all the subsequent stochastic evolution of δ as a function of S, which in general results in trajectories going multiple times above and below the threshold, is irrelevant, since it corresponds to smaller-scale structures that will be erased and engulfed by the collapse and virialization of the halo corresponding to the largest value of R, i.e. the smallest value of S, for which the threshold has been crossed. In other words, trajectories should be eliminated from further consideration once they have reached the threshold for the first time. In Bond et al. (1991) this is implemented by imposing the boundary condition

$$\Pi(\delta, S)|_{\delta=\delta_c} = 0 . \tag{10}$$

The solution of the FP equation with this boundary condition is

$$\Pi(\delta, S) = \frac{1}{\sqrt{2\pi S}}\left[e^{-\delta^2/(2S)} - e^{-(2\delta_c-\delta)^2/(2S)}\right] , \tag{11}$$

and gives the distribution function of excursion set theory. The first term is the PS result, while the second term in eq. (11) is an "image" Gaussian centered in $\delta = 2\delta_c$. Integrating this $\Pi(\delta, S)$ over $d\delta$ from $-\infty$ to δ_c gives the probability that a trajectory, at "time" S, has always been below the threshold. Increasing S this integral decreases because more and more trajectories cross the threshold for the first time, so the probability of first crossing the threshold between "time" S and $S + dS$ is given by $\mathcal{F}(S)dS$, with

$$\mathcal{F}(S) = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c} d\delta\, \Pi(\delta; S) . \tag{12}$$

With standard manipulations (see e.g. Zentner (2007) or MR1) one then finds that the function $f(\sigma)$ which appears in eq. (5) is given by

$$f(\sigma) = 2\sigma^2\mathcal{F}(\sigma^2) , \tag{13}$$

where we wrote $S = \sigma^2$. Using eq. (11) one finds the PS prediction for the function $f(\sigma)$,

$$\begin{aligned} f_{\rm PS}(\sigma) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{\delta_c}{\sigma}\, e^{-\delta_c^2/(2\sigma^2)} \\ &= \left(\frac{2}{\pi}\right)^{1/2} \frac{\delta_c}{S^{1/2}}\, e^{-\delta_c^2/(2S)} , \end{aligned} \tag{14}$$

Observe that, when computing the first-crossing rate, the contribution of the Gaussian centered in $\delta = 0$ and of the image Gaussian in eq. (11) add up, giving the well-known factor of two that was missed in the original PS theory.

2.2 Path integral formulation of excursion set theory

While excursion set theory is quite elegant, and gives a first analytic understanding of the halo mass function, it suffers of two important set of problems. First, it is based on the spherical collapse model, which is, as we already mentioned, a significant oversimplification of the actual complex dynamics of halo formation. The second set of problems of excursion set theory is of a more technical nature, and is due to the fact that the Langevin equation with Dirac-delta noise, which is at the basis of the whole construction, can only be derived if one works with a sharp filter in *momentum* space, and if the fluctuations are Gaussian. However, as it is well known (Bond et al. 1991), and as we have discussed at length in MR1, with such a filter it is difficult to associate a halo mass to the smoothing scale R. When one uses a sharp filter in coordinate space, the evolution of the density with the smoothing scale becomes non-Markovian, and the corresponding first-passage time problem is technically much more difficult. In particular, the distribution function $\Pi(\delta, S)$ no longer satisfies a local differential equation such as the FP equation. The issue is particularly relevant when one wants to include non-Gaussianities in the formalism, since the inclusion of non-Gaussianities renders again the dynamics non-Markovian. Neglecting the non-Markovian dynamics

due to the filter function would lead to incorrectly assigning to non-Gaussianities in the primordial density field effects which are rather due, more trivially, to the procedure that one has adopted for smoothing the density field.

In MR1,MR3 has been developed a formalism that allows us to generalize excursion set theory to the case of a non-Markovian dynamics, either generated by the filter function or by primordial non-Gaussianities. The basic idea is the following. Rather than trying to derive a simple, local, differential equation for $\Pi(\delta, S)$ (which, as shown in MR1, is impossible; in the non-Markovian case $\Pi(\delta, S)$ rather satisfies a very complicated equation which is non-local with respect to "time" S), we construct the probability distribution $\Pi(\delta, S)$ directly by summing over all paths that never exceeded the threshold δ_c, i.e. by writing $\Pi(\delta, S)$ as a path integral with boundaries. To obtain such a representation, we consider an ensemble of trajectories all starting at $S_0 = 0$ from an initial position $\delta(0) = \delta_0$ and we follow them for a "time" S. We discretize the interval $[0, S]$ in steps $\Delta S = \epsilon$, so $S_k = k\epsilon$ with $k = 1, \ldots n$, and $S_n \equiv S$. A trajectory is then defined by the collection of values $\{\delta_1, \ldots, \delta_n\}$, such that $\delta(S_k) = \delta_k$. The probability density in the space of trajectories is

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) \equiv \langle \delta_D(\delta(S_1)-\delta_1) \ldots \delta_D(\delta(S_n)-\delta_n) \rangle \,, \quad (15)$$

where δ_D denotes the Dirac delta. Then the probability of arriving in δ_n in a "time" S_n, starting from an initial value δ_0, without ever going above the threshold, is[1]

$$\Pi_\epsilon(\delta_0; \delta_n; S_n) \equiv \int_{-\infty}^{\delta_c} d\delta_1 \ldots \int_{-\infty}^{\delta_c} d\delta_{n-1} \times W(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n). \quad (16)$$

The label ϵ in Π_ϵ reminds us that this quantity is defined with a finite spacing ϵ, and we are finally interested in the continuum limit $\epsilon \to 0$. As discussed in MR1 and MR3 (see Eqs. (23)-(27) and discussion therein), $W(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n)$ can be expressed in terms of the connected correlators of the theory,

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int \mathcal{D}\lambda \, e^Z \,, \quad (17)$$

where

$$\int \mathcal{D}\lambda \equiv \int_{-\infty}^{\infty} \frac{d\lambda_1}{2\pi} \ldots \frac{d\lambda_n}{2\pi} \,, \quad (18)$$

and

$$Z = i \sum_{i=1}^n \lambda_i \delta_i + \sum_{p=2}^{\infty} \frac{(-i)^p}{p!} \sum_{i_1=1}^n \ldots \sum_{i_p=1}^n \lambda_{i_1} \ldots \lambda_{i_p} \langle \delta_{i_1} \ldots \delta_{i_p} \rangle_c \,. \quad (19)$$

We also used the notation $\delta_i = \delta(S_i)$, and $\langle \delta_1 \ldots \delta_n \rangle_c$ denotes the connected n-point correlator. So

$$\Pi_\epsilon(\delta_0; \delta_n; S_n) = \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \int \mathcal{D}\lambda \, e^Z \,. \quad (20)$$

[1] In eqs. (9) and (11) we had implicitly assumed $\delta_0 = 0$. In the following however it will be necessary to keep track also of the initial position δ_0.

When $\delta(S)$ satisfies eqs. (6) and (7) (which is the case for sharp filter in wavenumber space) the two-point function can be easily computed, and is given by

$$\langle \delta(S_i) \delta(S_j) \rangle = \min(S_i, S_j) \,. \quad (21)$$

If furthermore we consider Gaussian fluctuations, all n-point connected correlators with $n \geqslant 3$ vanish, and the probability density W can be computed explicitly,

$$W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \frac{1}{(2\pi\epsilon)^{n/2}} \, e^{-\frac{1}{2\epsilon} \sum_{i=0}^{n-1} (\delta_{i+1} - \delta_i)^2} , \quad (22)$$

where the superscript "gm" (Gaussian-Markovian) reminds us that this value of W is computed for Gaussian fluctuations, and when the evolution with respect to the smoothing scale is Markovian. Using this result, in MR1 we have shown that, in the continuum limit, the distribution function $\Pi_{\epsilon=0}(\delta; S)$, computed with a sharp filter in wavenumber space, satisfies a Fokker-Planck equation with the boundary condition $\Pi_{\epsilon=0}(\delta_c, S) = 0$, and we have therefore recovered, from a path integral approach, the distribution function of excursion set theory, eq. (11). Considering a more realistic filter, such as a step function in coordinate space, necessarily introduces non-Markovianity and the computation, which is quite non-trivial from a technical point of view, has been discussed in great detail in MR1. In order to make the computation of the first-crossing rate with a moving barrier more clear, from now on we will adopt the step function in wavenumber space as a filter and eliminate the source of non-Markovianity given by the choice of the window function. The effect of a more realistic filter function could then be computed as in MR1. The effect, however, will be tiny and totally negligible in the large mass range we are mostly interested in for the non-Gaussian case. Let us just close this subsection by reminding the reader about some useful properties of the path integral formulation which will turn out to be useful in the following. We will encounter objects such as

$$\sum_{i=1}^{n-1} F(S_i) \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \, \partial_i W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) \,, \quad (23)$$

where F denotes a generic function. To compute this expression we integrate ∂_i by parts,

$$\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \, \partial_i W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int_{-\infty}^{\delta_c} d\delta_1 \ldots \widehat{d\delta_i} \ldots d\delta_{n-1} \times W(\delta_0; \delta_1, \ldots, \delta_i = \delta_c, \ldots, \delta_{n-1}, \delta_n; S_n) \,, \quad (24)$$

where the notation $\widehat{d\delta_i}$ means that we must omit $d\delta_i$ from the list of integration variables. We next observe that $W^{\rm gm}$ satisfies

$$W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_i = \delta_c, \ldots, \delta_n; S_n) = W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_{i-1}, \delta_c; S_i) \times W^{\rm gm}(\delta_c; \delta_{i+1}, \ldots, \delta_n; S_n - S_i) \,, \quad (25)$$

as can be verified directly from its explicit expression (22). Then

$$\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{i-1} \int_{-\infty}^{\delta_c} d\delta_{i+1} \ldots d\delta_{n-1}$$

$$\times W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_{i-1},\delta_c;S_i)W^{\rm gm}(\delta_c;\delta_{i+1},\ldots,\delta_n;S_n-S_i)$$
$$=\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S_i)\Pi_\epsilon^{\rm gm}(\delta_c;\delta_n;S_n-S_i)\,, \qquad (26)$$

and to compute the expression given in eq. (23) we must compute objects such as

$$\sum_{i=1}^{n-1}F(S_i)\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S_i)\Pi_\epsilon^{\rm gm}(\delta_c;\delta_n;S_n-S_i). \qquad (27)$$

To proceed further, we need to know $\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S_i)$. By definition, for $\epsilon=0$ this quantity vanishes, since its second argument is equal to the the threshold value δ_c, compare with eq. (10). However, in the continuum limit the sum over i becomes $1/\epsilon$ times an integral over an intermediate time variable S_i,

$$\sum_{i=1}^{n-1}\to\frac{1}{\epsilon}\int_o^{S_n}dS_i\,, \qquad (28)$$

so we need to know how $\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S_i)$ approaches zero when $\epsilon\to 0$. In MR1 we proved that it vanishes as $\sqrt{\epsilon}$, and that

$$\Pi_\epsilon^{\rm gm}(\delta_0;\delta_c;S)=\sqrt{\epsilon}\,\frac{1}{\sqrt{\pi}}\,\frac{\delta_c-\delta_0}{S^{3/2}}e^{-(\delta_c-\delta_0)^2/(2S)}+\mathcal{O}(\epsilon)\,. \quad (29)$$

Similarly, for $\delta_n<\delta_c$,

$$\Pi_\epsilon^{\rm gm}(\delta_c;\delta_n;S)=\sqrt{\epsilon}\,\frac{1}{\sqrt{\pi}}\,\frac{\delta_c-\delta_n}{S^{3/2}}e^{-(\delta_c-\delta_n)^2/(2S)}+\mathcal{O}(\epsilon)\,. (30)$$

In the following, we will also need the expression for $\Pi_\epsilon^{\rm gm}$ with the first and second argument both equal to δ_c, which is given by (see again MR1)

$$\Pi_\epsilon^{\rm gm}(\delta_c;\delta_c;S)=\frac{\epsilon}{\sqrt{2\pi}S^{3/2}}\,. \qquad (31)$$

The two factors $\sqrt{\epsilon}$ from eqs. (29) and (30) produce just an overall factor of ϵ that compensates the factor $1/\epsilon$ in eq. (28), and we are left with a finite integral over dS_i. Terms with two or more derivative, e.g. $\partial_i\partial_j$, or $\partial_i,\partial_j\partial_k$ acting on W, with all indices i,j,k maller than n, can be computed similarly, and have been discussed in detail in MR1. With these technical details in mind, one can proceed to the computation of the first-crossing rate in the presence of a moving barrier.

3 PATH INTEGRAL WITH MOVING BARRIER: GAUSSIAN FLUCTUATIONS AND MARKOVIAN EVOLUTION WITH THE SMOOTHING SCALE

In this section we discuss the first-crossing rate for a generic moving barrier $B(S)$, specializing to the ellipsoidal one at the end. We consider first the case of Gaussian primordial fluctuations, and we will assume that the evolution with the smoothing scale is Markovian. Similarly to the constant barrier case, the probability of arriving at δ_n in a "time" S_n, starting from the initial value $\delta_0=0$, without ever going above the threshold, is

$$\Pi_\epsilon(\delta_n;S_n)\;\equiv\;\int_{-\infty}^{B(S_1)}d\delta_1\ldots\int_{-\infty}^{B(S_{n-1})}d\delta_{n-1} \qquad (32)$$
$$\times W(\delta_0;\delta_1,\ldots,\delta_{n-1},\delta_n;S_n).$$

Since we are considering the Gaussian and Markovian case, $W(\delta_0;\delta_1,\ldots,\delta_{n-1},\delta_n;S_n)$ can be expressed in terms of the connected two-point function of the theory, as

$$W(\delta_0;\delta_1,\ldots,\delta_n;S_n)=\int\mathcal{D}\lambda$$
$$\times\exp\left\{i\sum_{i=1}^n\lambda_i\delta_i-\frac{1}{2}\sum_{i,j=1}^n\lambda_i\lambda_j\min(S_i,S_j)\right\}. \qquad (33)$$

Taking the derivative with respect to the time $S_n\equiv S$ of eq. (32) and using the fact that $i\lambda_j$ $(j=1,\cdots,n)$ can be replaced ∂_j, we discover that $\Pi_\epsilon(\delta;S)$ satisfies the Fokker-Planck (FP) equation

$$\frac{\partial\Pi_\epsilon(\delta;S)}{\partial S}=\frac{1}{2}\frac{\partial^2\Pi_\epsilon(\delta;S)}{\partial\delta^2}, \qquad (34)$$

(where we used the notation $\delta_n=\delta$). To determine the boundary condition to be imposed on the solution of eq. (34) we proceed as follows. We start from eq. (32), with W given by eq. (22) and, shifting the variables δ_i $(i=1,\ldots,n)$ as $\delta_i\to\delta_i-B(S_i)$, we obtain

$$\Pi_\epsilon(\delta_n+B_n;S_n)=\int_{-\infty}^0d\delta_1\ldots\int_{-\infty}^0d\delta_{n-1}$$
$$\times\frac{1}{(2\pi\epsilon)^{n/2}}e^{-\frac{1}{2\epsilon}\sum_{i=0}^{n-1}[\delta_{i+1}-\delta_i+B_n-B_{n-1}]^2}$$
$$=\int_{-\infty}^0d\delta_{n-1}\frac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{1}{2\epsilon}[\delta_n-\delta_{n-1}+B_n-B_{n-1}]^2}$$
$$\times\Pi_\epsilon(\delta_{n-1}+B_{n-1};S_{n-1})\,, \qquad (35)$$

where we used the notation $B_i\equiv B(S_i)$, so $B_n\equiv B_n$. Now let $S_{n-1}=S$ so $S_n=S+\epsilon$, and $\delta_n+B(S)=\delta,\delta_n-\delta_{n-1}=\Delta\delta$. For fixed δ_n, we have $d\delta_{n-1}=-d(\Delta\delta)$. By further taking the limit $\epsilon\to 0$ (assuming that $B(S)$ is a continuous and differentiable function), eq. (35) becomes

$$\Pi_{\epsilon=0}(\delta;S)=\int_{\delta-B(S)}^{\infty}d(\Delta\delta)\delta_D(\Delta\delta)\Pi_{\epsilon=0}(\delta-\Delta\delta;S)\,. \qquad (36)$$

From this relation we get the boundary condition. If $\delta=B(S)$ the integral is over half of the support of the Dirac delta and so $\Pi_{\epsilon=0}(B(S);S)=(1/2)\Pi_{\epsilon=0}(B(S);S)$ hence $\Pi_{\epsilon=0}(B(S);S)=0$. Furthermore, if $\delta>B(S)$, the support of the Dirac delta is outside the integration limits and therefore we conclude that

$$\Pi_{\epsilon=0}(\delta;S)=0\qquad\text{for}\quad\delta\geqslant B(S)\,. \qquad (37)$$

In the continuum limit the first-crossing rate is then given by

$$\mathcal{F}(S)\;=\;-\frac{\partial}{\partial S}\int_{-\infty}^{B(S)}d\delta\,\Pi_{\epsilon=0}(\delta;S) \qquad (38)$$
$$=\;-\frac{dB(S)}{dS}\Pi_{\epsilon=0}(B(S),S)-\int_{-\infty}^{B(S)}d\delta\,\frac{\partial\Pi_{\epsilon=0}(\delta;S)}{\partial S}\,.$$

The first term on the right-hand side vanishes because of the boundary condition, while the second term can be written in a more convenient form using the FP equation (34), so

$$\mathcal{F}(S)\;\;=\;\;-\frac{1}{2}\int_{-\infty}^{B(S)}d\delta\,\frac{\partial^2\Pi_{\epsilon=0}(\delta;S)}{\partial\delta^2}$$

$$= \quad -\frac{1}{2}\left.\frac{\partial \Pi_{\epsilon=0}(\delta;S)}{\partial \delta}\right|_{\delta=B(S)}. \tag{39}$$

To compute the probability $\Pi_{\epsilon=0}(\delta_n, S_n)$ we proceed in the following way. At every i-th step of the path integral we Taylor expand the barrier around its final value

$$B(S_i) = B(S_n) + \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!}\,(S_i - S_n)^p\,, \tag{40}$$

where

$$B_n^{(p)} \equiv \frac{d^p B(S_n)}{dS_n^p}\,, \tag{41}$$

(so in particular $B_n^{(0)} = B(S_n) \equiv B_n$). We now perform a shift in the variable δ_i $(i = 1,\ldots,n-1)$ in the path integral

$$\delta_i \to \delta_i - \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!}\,(S_i - S_n)^p\,, \tag{42}$$

Then $\Pi_\epsilon(\delta_n; S_n)$ can be written as

$$\Pi_\epsilon(\delta_n; S_n) = \int_{-\infty}^{B_n} d\delta_1 \ldots \int_{-\infty}^{B_n} d\delta_{n-1} \int \mathcal{D}\lambda\, e^Z \tag{43}$$

where

$$\begin{aligned} Z \quad = \quad & i\sum_{i=1}^{n} \lambda_i\delta_i - \frac{1}{2}\sum_{i,j=1}^{n} \lambda_i\lambda_j \min(S_i,S_j) \\ & + i\sum_{i=1}^{n-1}\lambda_i \sum_{p=1}^{\infty}\frac{B_n^{(p)}}{p!}\,(S_i-S_n)^p\,. \end{aligned} \tag{44}$$

We next expand

$$\begin{aligned} & \exp\left\{ i\sum_{i=1}^{n-1}\lambda_i \sum_{p=1}^{\infty}\frac{B_n^{(p)}}{p!}\,(S_i-S_n)^p \right\} \\ & \simeq 1 + i\sum_{i=1}^{n-1}\lambda_i \sum_{p=1}^{\infty}\frac{B_n^{(p)}}{p!}\,(S_i-S_n)^p \\ & -\frac{1}{2}\sum_{i,j=1}^{n-1}\lambda_i\lambda_j \sum_{p,q=1}^{\infty}\frac{B_n^{(p)}B_n^{(q)}}{p!q!}\,(S_i-S_n)^p\,(S_j-S_n)^q + \cdots\,, \end{aligned} \tag{45}$$

and we write $\Pi_\epsilon(\delta_n; S_n)$ as

$$\begin{aligned} \Pi_\epsilon(\delta_n; S_n) \quad = \quad & \Pi_\epsilon^{(0)}(\delta_n; S_n) + \Pi_\epsilon^{(1)}(\delta_n; S_n) \\ & + \Pi_\epsilon^{(2)}(\delta_n; S_n) + \cdots\,, \end{aligned} \tag{46}$$

where

$$\Pi_{\epsilon=0}^{(0)}(\delta_n; S_n) = \frac{1}{\sqrt{2\pi S_n}}\left[e^{-\delta_n^2/(2S_n)} - e^{-(2B_n-\delta_n)^2/(2S_n)}\right], \tag{47}$$

$$\begin{aligned} \Pi_\epsilon^{(1)}(\delta_n; S_n) \quad = \quad & \sum_{i=1}^{n-1}\int_{-\infty}^{B_n} d\delta_1 \ldots d\delta_{n-1} \sum_{p=1}^{\infty}\frac{B_n^{(p)}}{p!} \\ & \times (S_i-S_n)^p\,\partial_i W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n)\,, \end{aligned} \tag{48}$$

and

$$\begin{aligned} & \Pi_\epsilon^{(2)}(\delta_n; S_n) = \frac{1}{2}\sum_{i,j=1}^{n-1}\int_{-\infty}^{B_n} d\delta_1 \ldots d\delta_{n-1} \sum_{p,q=1}^{\infty}\frac{B_n^{(p)}B_n^{(q)}}{p!q!} \\ & \times (S_i-S_n)^p\,(S_j-S_n)^q\,\partial_i\partial_j W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n)\,. \end{aligned} \tag{49}$$

We have therefore formally expanded $\Pi_{\epsilon=0}(\delta_n, S_n)$ in a series of terms $\Pi_{\epsilon=0}^{(1)}$, $\Pi_{\epsilon=0}^{(2)}$, etc., in which each term is itself given by an infinite sum over indices $p, q, \ldots$. To proceed further, we must either perform some approximation, or identify a suitable small parameter, and organize the terms in a systematic expansion in such a small parameter. In the next subsections we first discuss the approximation in which one can rederive the Sheth-Tormen result, and we will then compare it with two complementary, and more systematic, expansions.

3.1 The Sheth-Tormen approximation

To attack the problem, a first idea is to perform the integrals in eqs. (48) and (49) approximating $(S_n - S_i)^{p-1} \simeq S_n^{p-1}$ inside the integrals. This is in fact equivalent to the approximation made in Lam & Sheth (2009), see in particular their eq. (20). The detailed calculations, within our formalism, are reported in Appendix A and one obtains the first-crossing rate for a moving barrier

$$\mathcal{F}_{\rm ST}(S) \quad = \quad \frac{e^{-B^2(S)/(2S)}}{\sqrt{2\pi}S^{3/2}}\sum_{p=0}^{\infty}\frac{(-S)^p}{p!}\frac{\partial^p B(S)}{\partial S^p}\,. \tag{50}$$

This expression agrees with the one suggested in Sheth & Tormen (2002). Notice that for the cases of constant barrier $B(S) = \delta_c$ and of a linear barrier $B(S) = \delta_c + \beta S$, which are the known examples where the first-crossing rate can be computed analytically by solving exactly the FP equation in the presence of such a barrier (for the linear barrier see Sheth (1998) and Section IX of Zentner (2007)) the first-crossing rate (50) reproduces the correct answer. When applied to the ellipsoidal barrier given in eq. (2), and restricting the sum to $p \leqslant 5$, one recovers the ellipsoidal collapse result of Sheth & Tormen (2002)

$$\begin{aligned} \mathcal{F}_{\rm ST}^{\rm ell}(S) \quad \simeq \quad & \frac{\sqrt{a}\,\delta_c}{\sqrt{2\pi}S^{3/2}}e^{-B^2(S)/(2S)}\Bigg[1 + \\ & +0.4\sum_{p=0}^{5}(-1)^p\binom{0.6}{p}\left(\frac{S}{a\delta_c^2}\right)^{0.6}\Bigg] \\ = \; & \frac{\sqrt{a}\,\delta_c}{\sqrt{2\pi}S^{3/2}}e^{-B^2(S)/(2S)}\left[1 + 0.067\left(\frac{S}{a\delta_c^2}\right)^{0.6}\right]. \end{aligned} \tag{51}$$

This procedure is, however, not free from drawbacks. Indeed, the restriction of the sum to $p \leqslant 5$ is not justified and is merely dictated by the fact that stopping arbitrarily the series at $p = 5$ provides a a good fit to the N-body simulations.[2] However, if the sum over p is extended up to infinity the sum simply resums to $B(0)$ since, performing a Taylor expansion of $B(S_0 - S)$ in powers of S and setting finally $S_0 = S$, we have

$$B(0) = \sum_{p=0}^{\infty}\frac{(-S)^p}{p!}\frac{\partial^p B(S)}{\partial S^p}\,. \tag{52}$$

Since $B(0) = \sqrt{a}\delta_c$, we just end up with

$$\mathcal{F}_{p=\infty}^{\rm ell}(S) = \frac{\sqrt{a}\,\delta_c}{\sqrt{2\pi}S^{3/2}}e^{-B^2(S)/(2S)}\,, \tag{53}$$

[2] We thank Ravi Sheth for discussions about this point.

so the correction term $\sim S^{0.6}$ in eq. (51) seems an artifact of stopping the sum to $p = 5$. This is a rather puzzling result, since this correction is known to fit well the data, and is widely used in the literature. This calls for a different and more rigorous approach where the integrals are performed without the approximation $(S_n - S_i)^{p-1} \simeq S_n^{p-1}$. We discuss two different possible approaches in the next two subsection.

3.2 Expansion of $\Pi_\epsilon(\delta, S)$ in derivatives of $B(S)$

In order to develop a more systematic expansion, we first consider the case of a barrier $B(S)$ which is slowly varying with S. In this case, the small parameters are the derivatives of the function $B(S)$.

At first one might think that such an approximation, altough useful in some cases, would not apply to the barrier which corresponds to the the ellipsoidal collapse, eq. (2). In this case infact $B_{\rm el}(S)$ is given by a constant plus a term proportional to S^γ with $\gamma \simeq 0.6 < 1$, and therefore already its first derivative, which is proportional to $S^{\gamma-1}$ is large at sufficiently small S, and formally even diverges as $S \to 0$. However one should not forget that, in practice, even the largest galaxy clusters than one finds in observations, as well as in large-scale N-body simulations, have typical masses smaller than about $10^{15}h^{-1}M_\odot$ which, in the standard ΛCDM cosmology, corresponds to values of $S = \sigma^2(M) \gtrsim 0.35$, see e.g. Fig. 1 of Zentner (2007). Even for such a value, which is the smallest in which we are interested, the value of $B'_{\rm el}(S)$ is just of order 0.3 which means that, in the range of masses of interest, the barrier of ellipsoidal collapse can be considered as slowly varying.

We therefore expand $\Pi_\epsilon(\delta_n; S_n)$ in powers of the derivatives of the barrier, keeping terms with the same number of derivatives, so for instance a term proportional to d^2B/dS^2 is taken to be of the same order as $(dB/dS)^2$. Working up to terms of second order in the derivatives we get

$$\begin{aligned}\Pi_\epsilon(\delta_n; S_n) &= \Pi_\epsilon^{(0)}(\delta_n; S_n) + \Pi_\epsilon^{(a)}(\delta_n; S_n) + \Pi_\epsilon^{(b)}(\delta_n; S_n) \\ &\quad + \Pi_\epsilon^{(c)}(\delta_n; S_n)\,, \end{aligned} \tag{54}$$

where

$$\Pi_\epsilon^{(a)}(\delta_n; S_n) = \sum_{i=1}^{n-1} B'_n(S_i - S_n) \times \int_{-\infty}^{B_n} d\delta_1 \ldots d\delta_{n-1} \partial_i W^{\rm gm}\,, \tag{55}$$

$$\Pi_\epsilon^{(b)}(\delta_n; S_n) = \frac{1}{2}\sum_{i=1}^{n-1} B''_n(S_i - S_n)^2 \times \int_{-\infty}^{B_n} d\delta_1 \ldots d\delta_{n-1} \partial_i W^{\rm gm}\,, \tag{56}$$

$$\Pi_\epsilon^{(c)}(\delta_n; S_n) = \frac{1}{2}\sum_{i,j=1}^{n-1} \left(B'_n\right)^2 (S_i - S_n)(S_j - S_n) \times \int_{-\infty}^{B_n} d\delta_1 \ldots d\delta_{n-1} \partial_i \partial_j W^{\rm gm}, \tag{57}$$

and we used a prime to denote the derivatives of $B(S_n)$ with respect to S_n. Observe that $\Pi^{(a)}$ and $\Pi^{(b)}$ are linear in the first and second derivative, respectively, and come from the terms $p = 1, 2$ of $\Pi^{(1)}$, while $\Pi^{(c)}$ is quadratic in the first derivative, and is the term $p = q = 1$ of $\Pi^{(2)}$.

In Appendix B we compute these three terms, in the continuum limit, using the techniques developed in MR1. For the first term we find

$$\Pi_{\epsilon=0}^{(a)}(\delta_n; S_n) = -2B'_n \frac{(B_n - \delta_n)}{\sqrt{2\pi S_n}} e^{-(2B_n - \delta_n)^2/(2S_n)}\,. \tag{58}$$

Observe that it satisfies the boundary condition $\Pi_{\epsilon=0}^{(a)}(\delta_n; S_n) = 0$ when $\delta_n = B_n$, as it should. For the second term we get

$$\Pi_{\epsilon=0}^{(b)}(\delta_n; S_n) = \frac{1}{2\pi} B''_n (B_n - \delta_n) \times \left[\sqrt{2\pi S_n} e^{-(2B_n - \delta_n)^2/(2S_n)} - \pi B_n {\rm Erfc}\left(\frac{2B_n - \delta_n}{\sqrt{2S_n}}\right)\right], \tag{59}$$

and again vanishes linearly as $\delta_n \to B_n$. The third term is given by

$$\Pi_{\epsilon=0}^{(c)}(\delta_n; S_n) = -2\left(B'_n\right)^2 \frac{(B_n - \delta_n)^2}{\sqrt{2\pi S_n}} e^{-(2B_n - \delta_n)^2/(2S_n)}, \tag{60}$$

and vanishes quadratically as $\delta_n \to B_n$. This means that in the end it does not contribute to the first-crossing rate, since, using eq. (39), the latter is given by the derivative of $\Pi_{\epsilon=0}(\delta_n; S_n)$ with respect to δ_n, evaluated in $\delta_n = B_n$.

It is interesting to check explicitly that this solution for $\Pi(\delta_n; S_n)$ satisfies the FP equation, up to order to which we have computed, i.e. up to terms of second order in the derivatives of the barrier, included. Define the FP operator

$$\hat{D} = \frac{\partial}{\partial S_n} - \frac{1}{2}\frac{\partial^2}{\partial \delta_n^2}\,, \tag{61}$$

and define $f^{(0)}, \ldots f^{(c)}$ from

$$\hat{D}\Pi_{\epsilon=0}^{A}(\delta_n; S_n) = \sqrt{\frac{2}{\pi}}\frac{1}{S_n^{3/2}} e^{-(2B_n - \delta)^2/(2S_n)} f^A\,, \tag{62}$$

where $A = (0), (a), (b), (c)$ so, up to terms of second order (included) in the derivatives of the barrier,

$$\begin{aligned}\hat{D}\Pi_{\epsilon=0}(\delta_n; S_n) &= \sqrt{\frac{2}{\pi}}\frac{1}{S_n^{3/2}} e^{-(2B_n - \delta_n)^2/(2S_n)} \\ &\quad \times [f^{(0)} + f^{(a)} + f^{(b)} + f^{(c)}]\,. \end{aligned} \tag{63}$$

Inserting the expressions for $\Pi_{\epsilon=0}^{(0)}, \Pi_{\epsilon=0}^{(a)}, \Pi_{\epsilon=0}^{(b)}, \Pi_{\epsilon=0}^{(c)}$ computed above we get

$$f^{(0)} = (2B_n - \delta_n)B'_n\,, \tag{64}$$

$$\begin{aligned} f^{(a)} &= -(2B_n - \delta_n)B'_n - S_n(B_n - \delta_n)B''_n \\ &\quad + [2(B_n - \delta_n)(2B_n - \delta_n) - S_n](B'_n)^2 \end{aligned} \tag{65}$$

$$f^{(b)} = S_n(B_n - \delta_n)B''_n + \mathcal{O}(B'''_n, B'_nB''_n, (B'_n)^3) \tag{66}$$

$$\begin{aligned} f^{(c)} &= -[2(B_n - \delta_n)(2B_n - \delta_n) - S_n](B'_n)^2 \\ &\quad + \mathcal{O}(B'''_n, B'_nB''_n, (B'_n)^3) \end{aligned} \tag{67}$$

Therefore the sum $\Pi_{\epsilon=0}^{(0)} + \Pi_{\epsilon=0}^{(a)} + \Pi_{\epsilon=0}^{(b)} + \Pi_{\epsilon=0}^{(c)}$ satisfies the FP equation, modulo terms of third order in the derivative of the barrier.

The first-crossing rate is then readily evaluated through eq. (39). The zero-th order contribution from $\Pi_{\epsilon=0}^{(0)}$ is

$$\mathcal{F}^{(0)}(S) = \frac{B(S)}{\sqrt{2\pi}S^{3/2}} e^{-B^2(S)/(2S)}\,, \tag{68}$$

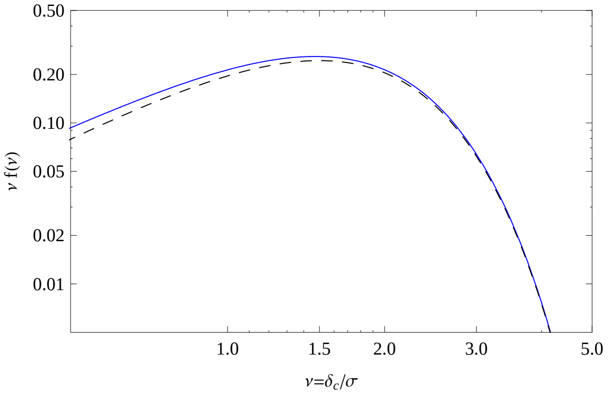

Figure 1. The Sheth-Tormen first-crossing rate for the ellipsoidal barrier $\mathcal{F}_{\rm ST}^{\rm ell}$ (dashed black line), compared to the first-crossing rate $\mathcal{F}^{(2)}_{\rm der}$ (solid blue line) obtained from the expansion in derivatives of the barrier, as a function of ν.

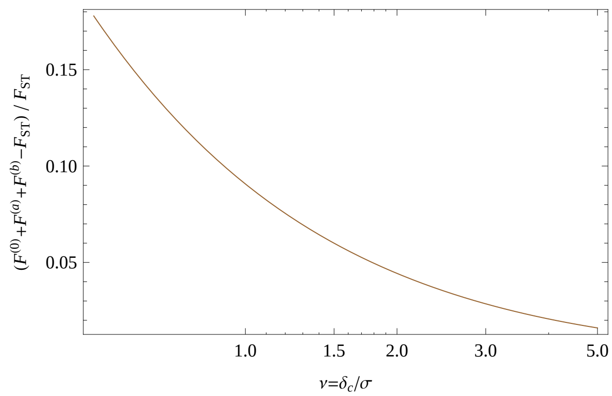

Figure 2. The ratio $(\mathcal{F}^{(2)}_{\rm der}-\mathcal{F}_{\rm ST})/\mathcal{F}_{\rm ST}$, as a function of ν.

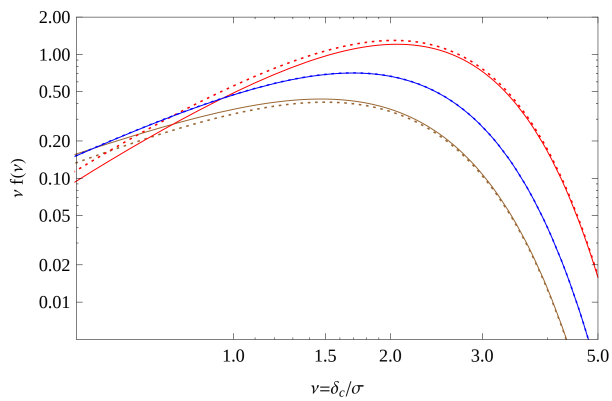

Figure 3. First-crossing rate for filaments (blue), sheets (red) and halos (brown). The dotted lines refer to the ST approximation (50) with $p\leqslant 5$, while the continuous ones refer to our result (71).

while the higher orders give

$$\mathcal{F}^{(a)}(S) = -\frac{B'(S)}{\sqrt{2\pi S}}e^{-B(S)^2/(2S)}\,, \tag{69}$$

$$\mathcal{F}^{(b)}(S) = \frac{B''(S)}{4\pi} \times\left\{\sqrt{2\pi S}e^{-B(S)^2/(2S)} - \pi B(S){\rm Erfc}\left[\frac{B(S)}{2S}\right]\right\}. \tag{70}$$

and $\mathcal{F}^{(c)}=0$, as already mentioned. In Fig. 1 we compare the Sheth-Tormen first crossing rate $\mathcal{F}_{\rm ST}(S)$ to the quantity

$$\mathcal{F}^{(2)}_{\rm der}(S)=\mathcal{F}^{(0)}(S)+\mathcal{F}^{(a)}(S)+\mathcal{F}^{(b)}(S)\,, \tag{71}$$

i.e. to the first crossing rate obtained by performing the expansion in derivatives of the barrier, up to second order (included) in the derivatives, while in Fig. 2 we plot the relative difference $(\mathcal{F}^{(2)}_{\rm der}-\mathcal{F}_{\rm ST})/\mathcal{F}_{\rm ST}$. We see that the two results agree perfectly at large values of ν (i.e. at large masses), and they still agree to better than 10% down to $\nu=1$.

The fact that the $\mathcal{F}^{(2)}_{\rm der}$ is numerically quite close to $\mathcal{F}_{\rm ST}$ provides a more satisfying derivation of the ST mass function, showing that the approximation $(S_n-S_i)^{p-1}\simeq S_n^{p-1}$, together with the truncation to $p=5$ of the series in eq. (50), in the end gives a simple analytic formula which is numerically quite close to the result of a derivation based on a systematic expansion.

For comparison, we also report in Figure 3 the first-crossing rate for filaments (blue), sheets (red) and halos (brown). The dashed lines refer to the ST approximation (50) with $p\leqslant 5$, while the continuous ones refer to our result (71).

3.3 Expansion of $\Pi_{\epsilon=0}(\delta_n,S_n)$ in powers of $(B_n-\delta_n)$

In this subsection we describe a different expansion scheme, which allows us to resum a large number of terms. The basic idea is that, even if the computation of the distribution function Π can be interesting by itself in a more general context (since the probability distribution of a random walk in the presence of a moving barrier is a problem interesting in its own right in statistical physics), for the computation of the halo mass function we are really interested only in the first-crossing rate. Then eq. (39) shows that, in the Gaussian and Markovian case, we only need the derivative $\partial\Pi/\partial\delta_n$ evaluated at $\delta_n=B_n$. As shown in eq. (37), $\Pi(\delta_n,S_n)$ vanishes at $\delta_n=B_n$, so its Taylor expansion around $\delta_n=B_n$ starts from a term linear in (δ_n-B), followed by terms of order $(\delta_n-B)^2$, etc. When we compute $\partial\Pi/\partial\delta_n$ in $\delta_n=B_n$, the terms quadratic and higher-order in (δ_n-B), give zero, so we do not need the full function Π, but only the term linear in (δ_n-B) in its Taylor expansion around $\delta_n=B_n$. This simplifies our task considerably.

We first compute the part linear in (δ_n-B_n) of $\Pi^{(1)}$. Using the results of the previous section, in particular eqs. (24), (29) and (30), $\Pi^{(1)}_\epsilon(\delta_n;S_n)$ can be rewritten as

$$\Pi^{(1)}_{\epsilon=0}(\delta_n;S_n) = \frac{B_n(B_n-\delta_n)}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}B_n^{(p)} \times\int_0^{S_n}dS_i\,\frac{(S_n-S_i)^{p-(3/2)}}{S_i^{3/2}} \times e^{-B_n^2/(2S_i)}e^{-(B_n-\delta_n)^2/[2(S_n-S_i)]}\,. \tag{72}$$

For $p=0,1$ this integral can be computed analytically, see appendix C, but for $p\geqslant 2$ we have not been able to compute it exactly. However, for our purposes it is sufficient to

observe that in this expression for $\Pi^{(1)}_{\epsilon=0}$ there is already a factor $(B_n-\delta_n)$ in front of the integral over dS_i, and the integral converges at $S_i=S_n$ for all $p\geqslant 1$, even if in the integrand we set $\delta_n=B_n$. Therefore

$$\Pi^{(1)}_{\epsilon=0}(\delta_n;S_n) = \frac{B_n(B_n-\delta_n)}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}B_n^{(p)} \times\int_0^{S_n}dS_i\,\frac{(S_n-S_i)^{p-(3/2)}}{S_i^{3/2}}e^{-B_n^2/(2S_i)} +\mathcal{O}(B_n-\delta_n)^2\,. \tag{73}$$

In appendix C we show that for $p=1$ this integral is elementary while for $p\geqslant 2$ it can be computed in terms of the confluent hypergeometric function $U(a,b,z)$. As a result,

$$\Pi^{(1)}_{\epsilon=0}(\delta_n;S_n)=\frac{\sqrt{2}}{\pi}\,\frac{B_n-\delta_n}{S_n^{1/2}}\,e^{-B_n^2/(2S_n)} \times\left[\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}B_n^{(p)}S_n^{p-1}\,\Gamma\left(p-\frac{1}{2}\right)U\left(p-1,\frac{1}{2},\frac{B_n^2}{2S_n}\right)\right] +\mathcal{O}(B_n-\delta_n)^2\,, \tag{74}$$

where the term $p=1$ can be written in a more elementary form using $U(0,b,z)=1$ and $\Gamma(1/2)=\sqrt{\pi}$. Along the same lines, we have also computed the generic m-th order ($m\geqslant 1$) of the expansion of $\Pi_{\epsilon=0}$ (see App. D), at the linear order in $B_n-\delta_n$, and it is given by

$$\Pi^{(m)}_{\epsilon=0} = \frac{(B_n-\delta_n)e^{-\frac{B_n^2}{2S_n}}}{m!\,2^{\frac{m}{2}-1}\pi^{\frac{3-m}{2}}}\sum_{p_1,\dots,p_m=1}^{\infty}(-1)^{\sum_{k=1}^m p_k+m+1} \times \frac{B_n^{(p_1)}\cdots B_n^{(p_m)}}{p_1!\cdots p_m!}c_{p_2,\dots,p_m}S_n^{\sum_{k=1}^m p_k-\frac{m}{2}-1} \times \Gamma\left(\sum_{k=1}^m p_k-\frac{m}{2}\right)U\left(\sum_{k=1}^m p_k-\frac{m+1}{2},\frac{1}{2},\frac{B_n^2}{2S_n}\right) + \mathcal{O}(B_n-\delta_n)^2, \tag{75}$$

where the coefficients $c_{p,q,\cdots}$ can be computed by the recursion relations (D10)-(D11). This expression is useful for numerical evaluation, but not very illuminating from an analytic point of view. So it can be useful to keep in mind that in the limit $2S_n\ll B_n^2$, i.e. for large halo masses, the confluent hypergeometric U function simplifies to

$$U\left(k,\frac{1}{2},\frac{B_n^2}{2S_n}\right)\simeq\left(\frac{2S_n}{B_n^2}\right)^k\left[1+\mathcal{O}\left(\frac{2S_n}{B_n^2}\right)\right]\,. \tag{76}$$

The total probability is given by $\Pi=\sum_{m=0}^{\infty}\Pi^{(m)}$. We have not been able to resum all the terms of the expansion, but the first few terms are sufficient for the first-crossing rate. In fact, the first-crossing rate is readily evaluated through eq. (39). The zero-th order contribution from $\Pi^{(0)}_{\epsilon=0}$ is given by eq. (68) while higher-order contributions $\mathcal{F}^{(m)}$ are obtained from $\Pi^{(m)}_{\epsilon=0}$ in eq. (75), and are easily evaluated numerically. In Fig. 4, we plot $\mathcal{F}^{(0)}$ (blue) and $\mathcal{F}^{(0)}+\mathcal{F}^{(1)}+\mathcal{F}^{(1)}+\cdots$ (red), for the ellipsoidal barrier given in eq. (2). We deduce that the sum for Π converges quickly and the terms after the second one contribute negligibly to the first-crossing rate. It is therefore an excellent approximation to consider the first-crossing rate for a generic moving barrier $B(S)$ as given by $\mathcal{F}^{(0)}+\mathcal{F}^{(1)}$, i.e.

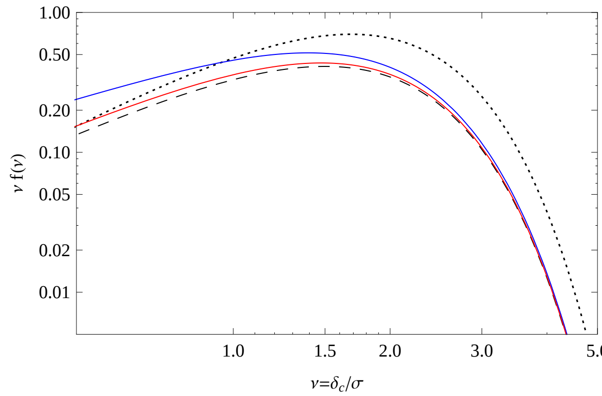

Figure 4. First-crossing rate for the ellipsoidal barrier (2). $\mathcal{F}_{\rm ST}$ (dashed black), $\mathcal{F}^{(0)}$ (solid blue), $\mathcal{F}^{(0)}+\mathcal{F}^{(1)}+\mathcal{F}^{(2)}+\cdots$ (solid red). The spherical collapse model, with the same value of $a=0.707$, corresponds to the dotted black line.

$$\mathcal{F}(S) = \frac{e^{-B^2(S)/(2S)}}{\sqrt{2\pi}S^{3/2}}\left[B(S) +\sum_{p=1}^{\infty}\frac{(-S)^p}{p!}\frac{\partial^p B(S)}{\partial S^p}\frac{\Gamma\left(p-\frac{1}{2}\right)}{\sqrt{\pi}}U\left(p-1,\frac{1}{2},\frac{B_n^2}{2S_n}\right)\right]\,. \tag{77}$$

For comparison, we also report in Fig. 4 the first-crossing rate of the spherical collapse model (dotted line) and the Sheth & Tormen (2002) result of eq. (51) (dashed line). Note also that eq. (77) reproduces the exact known results for the cases of constant and linear barrier shapes. It is also interesting to note that $\mathcal{F}(S)$ in eq. (77) and the rate $\mathcal{F}^{(2)}_{\rm der}(S)$ computed in the previous section differ by less than 5% for $\nu\geqslant 0.2$, for the ellipsoidal barrier (2). It is then reassuring to see that our two approaches to the computation of the first-crossing rate lead to consistent results, and their difference allows us to get a quantitative idea of the theoretical error in the computation. The fact that both results are numerically quite close to the ST mass function also provides a more satisfying justification of the ST mass function itself.

Armed with these results, we may now proceed to evaluate the halo mass function in the case in which non-Gaussianity (NG) is present.

4 THE ELLIPSOIDAL COLLAPSE AND NON-GAUSSIANITY

Deviations from Gaussianity are encoded, e.g., in the connected three- and four-point correlation functions which are dubbed the bispectrum and the trispectrum, respectively. A phenomenological way of parametrizing the level of NG is to expand the fully non-linear primordial Bardeen gravitational potential Φ in powers of the linear gravitational potential $\Phi_{\rm L}$

$$\Phi=\Phi_{\rm L}+f_{\rm NL}\left(\Phi_{\rm L}^2-\langle\Phi_{\rm L}^2\rangle\right)\,. \tag{78}$$

The dimensionless quantity $f_{\rm NL}$ sets the magnitude of the three-point correlation function (Bartolo et al. (2004)). If the process generating the primordial NG is local in space, the parameter $f_{\rm NL}$ in Fourier space is independent of the

momenta entering the corresponding correlation functions; if instead the process which generates the primordial cosmological perturbations is non-local in space, like in models of inflation with non-canonical kinetic terms, $f_{\rm NL}$ acquires a dependence on the momenta. The strongest current limits on the strength of local NG set the $f_{\rm NL}$ parameter to be in the range $-4 < f_{\rm NL} < 80$ at 95% confidence level (Smith et al 2010).

In MR3 the effect of primordial NG on the halo mass function was computed, using excursion set theory, for the case of a spherical collapse with constant barrier. In the presence of NG the stochastic evolution of the smoothed density field, as a function of the smoothing scale, is non-Markovian and beside "local" terms that generalize Press-Schechter (PS) theory, there are also "memory" terms, whose effect on the mass function have been computed using the formalism developed in MR1. When computing the effect of the three-point correlator on the mass function, a PS-like approach which consists in neglecting the cloud-in-cloud problem and in multiplying the final result by a fudge factor $\simeq 2$, is in principle not justified. Indeed, when computed correctly in the framework of excursion set theory, the "local" contribution vanishes (for all odd-point correlators the contribution of the image Gaussian cancels the Press-Schechter contribution rather than adding up), and the result comes entirely from non-trivial memory terms which are absent in PS theory. However it turns out that, in the limit of large halo masses, where the effect of non-Gaussianity is more relevant, these memory terms give a contribution which is the the same as that computed naively with PS theory, plus subleading terms depending on derivatives of the three-point correlator.

The goal of this section is to compute, using excursion set theory, the halo mass function in the presence of NG and for the ellipsoidal collapse, thus extending the findings of MR3 obtained for the spherical collapse. This computation is motivated by the fact that in the literature the halo mass function for the more realistic case of the ellipsoidal collapse is obtained, when NG is present, by multiplying the first-crossing rate (51) by a form factor $\mathcal{R}(f_{\rm NL}, S)$ obtained by dividing the first-crossing rates with and without NG for the PS spherical collapse case

$$\begin{aligned}\mathcal{F}_{\rm ST}(f_{\rm NL}, S) &= \mathcal{F}_{\rm ST}(f_{\rm NL}=0, S)\mathcal{R}(f_{\rm NL}, S)\\ &= \mathcal{F}_{\rm ST}(f_{\rm NL}=0, S)\frac{\mathcal{F}_{\rm PS}(f_{\rm NL}, S)}{\mathcal{F}_{\rm PS}(f_{\rm NL}=0, S)}\,. \end{aligned} \tag{79}$$

This procedure has however no rigourous justification and its validity should be tested with an explicit computation.

Similarly to the Gaussian case, the probability of arriving in δ_n in a "time" S_n, starting from the initial value $\delta_0 = 0$, without ever going above the threshold, in the presence of NG is given by

$$\begin{aligned}\Pi_\epsilon(\delta_n; S_n) \equiv& \int_{-\infty}^{B(S_1)} d\delta_1 \ldots \int_{-\infty}^{B(S_{n-1})} d\delta_{n-1}\\ &\times W_{\rm NG}(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n). \end{aligned} \tag{80}$$

where

$$W_{\rm NG}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int \mathcal{D}\lambda$$

$$\times \exp\left\{ i\sum_{i=1}^n \lambda_i\delta_i - \frac{1}{2}\sum_{i,j=1}^n \lambda_i\lambda_j \min(S_i, S_j)\right\}$$

$$\times \exp\left\{ \frac{(-i)^3}{6}\sum_{i,j,k=1}^n \langle\delta_i\delta_j\delta_k\rangle_c \lambda_i\lambda_j\lambda_k\right\}. \tag{81}$$

We now perform the shift (42) in the δ_i $(i = 1, \cdots, n-1)$ variables and expand the NG contribution to first order

$$\begin{aligned}\Pi_\epsilon(\delta_n; S_n) &= \Pi^{(0)}_{\epsilon=0}(\delta_n; S_n) + \Pi^{(1)}_{\epsilon=0}(\delta_n; S_n)\\ &+ \Pi^{(2)}_{\epsilon=0}(\delta_n; S_n) + \cdots\\ &- \frac{1}{6}\int_{-\infty}^{B_n} d\delta_1 \ldots \int_{-\infty}^{B_n} d\delta_{n-1} \sum_{i,j,k=1}^n\\ &\langle\delta_i\delta_j\delta_k\rangle_c \partial_i\partial_j\partial_k W_{\rm mb}(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n), \end{aligned} \tag{82}$$

where $W_{\rm mb}$ is the probability density in the space of trajectories with a moving barrier, so that

$$\begin{aligned}&\int_{-\infty}^{B_n} d\delta_1 \ldots \int_{-\infty}^{B_n} d\delta_{n-1}\, W_{\rm mb}(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n)\\ &= \Pi^{(0)}_{\epsilon=0} + \Pi^{(1)}_{\epsilon=0} + \Pi^{(2)}_{\epsilon=0} + \cdots\,. \end{aligned} \tag{83}$$

In principle the contribution from NG can be computed separating the various contributions to the sum according to whether an index is equal or smaller than n. In this way, however, the computations faces some technical difficulties. Fortunately, as discussed in MR3, the problem simplifies considerably in the limit of large halo masses, which is just the physically interesting limit. Large masses mean small values of S_n. The arguments S_i, S_j and S_k in the correlator $\langle\delta_i\delta_j\delta_k\rangle \equiv \langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle_c$ range over the interval $[0, S_n]$ and, if S_n goes to zero, we can expand the correlator in a multiple Taylor series around the point $S_i = S_j = S_k = S_n$. We introduce the notation

$$\begin{aligned}&G_3^{(p,q,r)}(S_n) \equiv\\ &\left[\frac{d^p}{dS_i^p}\frac{d^q}{dS_j^q}\frac{d^r}{dS_k^r}\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle_c\right]_{S_i=S_j=S_k=S_n}. \end{aligned} \tag{84}$$

Then

$$\begin{aligned}\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle &= \sum_{p,q,r=0}^{\infty} \frac{(-1)^{p+q+r}}{p!q!r!}(S_n - S_i)^p\\ &\times\; (S_n - S_j)^q (S_n - S_k)^r G_3^{(p,q,r,s)}(S_n)\,. \end{aligned} \tag{85}$$

The leading contribution to the halo mass function is given by the term in eq. (85) with $p = q = r = 0$ and we neglect subleading contributions, which can be computed with the same technique developed in MR3. The discrete sum reduces to $\langle\delta_n^3\rangle_c \sum_{i,j,k=1}^n \partial_i\partial_j\partial_k$ and we can split it as

$$\begin{aligned}\sum_{i,j,k=1}^n \partial_i\partial_j\partial_k &= \partial_n^3 + 3\sum_{i,j=1}^{n-1}\partial_i\partial_j\partial_n + 3\sum_{i=1}^{n-1}\partial_i\partial_n^2\\ &+ \sum_{i,j,k=1}^{n-1}\partial_i\partial_j\partial_k\,. \end{aligned} \tag{86}$$

When applying these derivatives to the $W_{\rm mb}$, one can use the identities proven in MR1 and MR3, namely

$$\sum_{i=1}^{n-1}\int_{-\infty}^{B_n} d\delta_1\ldots d\delta_{n-1}\,\partial_i W_{\rm mb}=\frac{\partial}{\partial B_n}\Pi_{\epsilon=0}\,, \tag{87}$$

$$\sum_{i,j=1}^{n-1}\int_{-\infty}^{B_n} d\delta_1\ldots d\delta_{n-1}\,\partial_i\partial_j W_{\rm mb}=\frac{\partial^2}{\partial B^2(S_n)}\Pi_{\epsilon=0}\,, \tag{88}$$

and

$$\sum_{i,j,k=1}^{n-1}\int_{-\infty}^{B_n} d\delta_1\ldots d\delta_{n-1}\,\partial_i\partial_j\partial_k W_{\rm mb}=\frac{\partial^3}{\partial B^3(S_n)}\Pi_{\epsilon=0}\,. \tag{89}$$

The probability density (82) calculated in this way vanishes at the barrier point $\delta_n=B_n$, when one properly expands the $\Pi_{\epsilon=0}$ according to one of the two methods described in the previous sections. This is a good check of the procedure we adopted and is necessary when evaluating the first-crossing rate.

The calculation of the first-crossing rate proceeds by integrating the probability density over δ_n and then taking the derivative with respect to S_n. This is fortunate because we can directly compute

$$\sum_{i,j,k=1}^{n}\int_{-\infty}^{B_n} d\delta_1\ldots d\delta_n\,\partial_i\partial_j\partial_k W_{\rm mb}
=\frac{\partial^3}{\partial B^3(S_n)}\int_{-\infty}^{B_n} d\delta_n \Pi_{\epsilon=0} \tag{90}$$

We choose two different expansions for Π. The expansion in derivatives of Sect. 3.2 gives

$$\frac{\partial^3}{\partial B^3(S_n)}\int_{-\infty}^{B_n} d\delta_n\,\left(\Pi^{(0)}_{\epsilon=0}+\Pi^{(a)}_{\epsilon=0}+\Pi^{(b)}_{\epsilon=0}+\Pi^{(c)}_{\epsilon=0}+\cdots\right)
=\frac{2}{\sqrt{2\pi}S^{5/2}}e^{-\frac{B^2}{2S}}\left[-S+B^2+SB'\left(B+2SB'\right)\right]
-\frac{3}{2}{\rm Erfc}\left[\frac{B}{\sqrt{2S}}\right]B''\,, \tag{91}$$

while the expansion using the approximation of Lam & Sheth (2009) (and discussed in Appendix A) gives

$$\frac{\partial^3}{\partial B^3(S_n)}\int_{-\infty}^{B_n} d\delta_n\,\left(\Pi^{(0)}_{\epsilon=0}+\Pi^{(1,{\rm ST})}_{\epsilon=0}+\Pi^{(2,{\rm ST})}_{\epsilon=0}+\cdots\right)
=-\sqrt{\frac{2}{\pi S_n^3}}\left(1-\frac{B^2(S_n)}{S_n}+\frac{B_n}{S_n}\mathcal{P}(S_n)-2\frac{\mathcal{P}^2(S_n)}{S_n}\right)
\times e^{-B^2(S_n)/(2S_n)}\,. \tag{92}$$

where

$$\mathcal{P}(S)\equiv\sum_{p=1}^{5}\frac{(-S)^p}{p!}\frac{\partial^p B(S)}{\partial S^p}\,. \tag{93}$$

Notice that the sum runs only up to $p=5$ to provide a good fit to the data, as mentioned earlier in Sect. 3.1. If we now normalize the bispectrum as

$$\mathcal{S}_3(S)\equiv\frac{1}{S^2}\langle\delta^3(S)\rangle\,, \tag{94}$$

we finally obtain the leading NG contribution to the first-crossing rate with a generic moving barrier. Using (91) we obtain

$$\mathcal{F}_{\rm NG}(S)=\mathcal{F}^{(0)}+\mathcal{F}^{(a)}+\mathcal{F}^{(b)}+\mathcal{F}^{(c)}
+\frac{\mathcal{S}_3}{12\sqrt{2\pi}S^{5/2}}\left[-2\left(S^2+2SB^2-B^4+SBB'\left(-7S+B^2\right)
-8S^3B'^2+4S^3BB'^3\right)+S^3B''\left(B+22SB'\right)\right]e^{-B^2/(2S)}
+\frac{S^2\mathcal{S}_3'}{3\sqrt{2\pi}S^{5/2}}\left[B^2+SBB'+S\left(-1+2SB'^2\right)\right]e^{-B^2/(2S)}
-\frac{S}{4}\left(\left(2\mathcal{S}_3+S\mathcal{S}_3'\right)B''+S\mathcal{S}_3B'''\right){\rm Erfc}\left[\frac{B}{\sqrt{2S}}\right]\,, \tag{95}$$

while using (92) we obtain

$$\mathcal{F}_{\rm NG}(S)=\frac{B+\mathcal{P}}{\sqrt{2\pi}S^{3/2}}e^{-B^2/(2S)}
+\frac{\mathcal{S}_3}{6\sqrt{2\pi}S^{5/2}}\left[B^4-B^3(\mathcal{P}+2SB')+2B^2(-S+\mathcal{P}^2
+S\mathcal{P}B')+SB(\mathcal{P}+6SB'-4\mathcal{P}^2B'-2S\mathcal{P}')
-S(S+2\mathcal{P}(\mathcal{P}+SB'-4S\mathcal{P}'))\right]e^{-B^2/(2S)}
+\frac{S^2\mathcal{S}_3'}{3\sqrt{2\pi}S^{5/2}}\left[B^2-B\mathcal{P}-S+2\mathcal{P}^2\right]e^{-B^2/(2S)}\,, \tag{96}$$

where the prime denotes differentation with respect to S.

Both formulae (95)-(96) can be further improved using a saddle-point technique in order to resum the largest contributions from NG, as in D'Amico et al. (2010). Limiting this procedure to the leading terms of (96) and treating $\mathcal{P}(S)$ and the derivatives of $B(S)$ as small parameters, we find for instance

$$\mathcal{F}_{\rm NG}(S)=\frac{Be^{-\frac{B^2}{2S}}}{\sqrt{2\pi}S^{3/2}}e^{\frac{1}{6}\mathcal{S}_3\frac{B^3}{S}}\left(1-\frac{1}{3}\mathcal{S}_3B-\frac{1}{6}\frac{S\mathcal{S}_3}{B}\right)
+\frac{\mathcal{P}}{\sqrt{2\pi}S^{3/2}}e^{-B^2/(2S)}
+\frac{\mathcal{S}_3}{6\sqrt{2\pi}S^{5/2}}\left[-B^3(\mathcal{P}+2SB')+2B^2(\mathcal{P}^2
+S\mathcal{P}B')+SB(\mathcal{P}+6SB'-4\mathcal{P}^2B'-2S\mathcal{P}')
-2S\mathcal{P}(\mathcal{P}+SB'-4S\mathcal{P}')\right]e^{-B^2/(2S)}
+\frac{S^2\mathcal{S}_3'}{3\sqrt{2\pi}S^{5/2}}\left[B^2-B\mathcal{P}-S+2\mathcal{P}^2\right]e^{-B^2/(2S)}\,. \tag{97}$$

Notice that, in the limit of constant barrier, our formulae are slightly different from those of D'Amico et al. (2010); we believe that the origin of this difference is due to the fact that they assumed a very specific form for the cumulants $\langle\delta_i\delta_j\delta_k\rangle\propto(S_iS_jS_k)^{1/2}$. With this assumption, one can find relations between the various derivatives of the cumulants, which otherwise are independent.

In the limit of constant barrier $B(S)=\sqrt{a}\delta_c$ one recovers the spherical collapse result of MR3 (neglecting the terms proportional to $\mathcal{S}_3'$)

$$\mathcal{F}^{\rm sph}_{\rm NG}(S)\ =\ \frac{\sqrt{a}\delta_c}{\sqrt{2\pi}S^{3/2}}e^{-a\delta_c^2/(2S)}\left[1+
\frac{S\,\mathcal{S}_3}{6\sqrt{a}\delta_c}\left(\frac{(\sqrt{a}\delta_c)^4}{S^2}-2\frac{(\sqrt{a}\delta_c)^2}{S}-1\right)\right]\,. \tag{98}$$

In Figure 5 we show the first-crossing rates (95) and (96), applied to the case of the ellipsoidal barrier (2). The two curves differ by $\mathcal{O}(10)\%$ at most in the small halo mass regime. In

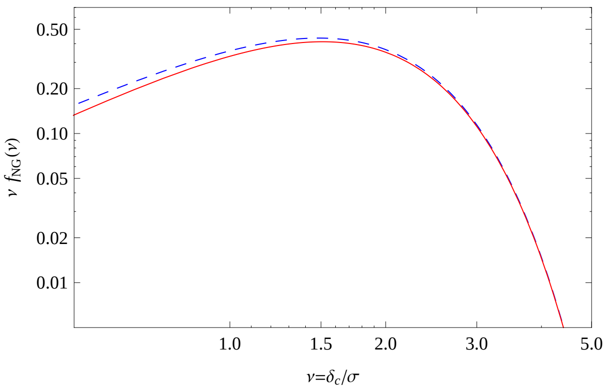

Figure 5. The first-crossing rate deduced from Eqs. (95) (dashed blue line) and (96) (solid red line), for the case of ellipsoidal barrier (2). We used $\mathcal{S}_3$ given by Eq. (101) with local $f_{\rm NL} = 100$.

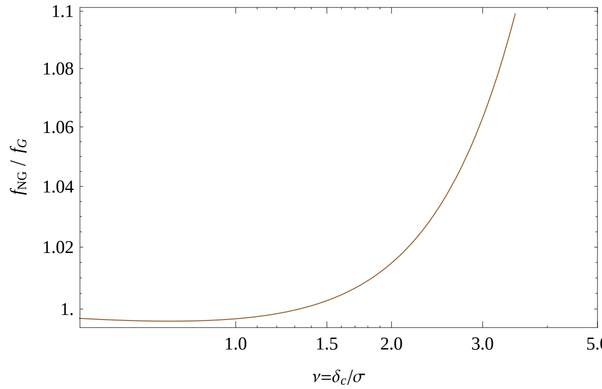

Figure 6. The ratio between the non-Gaussian first-crossing rate $f_{\rm NG}$ deduced from Eqs. (96) and the Gaussian one $f_{\rm G}$. We used $\mathcal{S}_3$ given by Eq. (101) with local $f_{\rm NL} = 100$.

Figure 6 we plot the ratio between the non-Gaussian first-crossing rate deduced from Eqs. (95) and the Gaussian one. In Figure 7 we show the ratios between the first-crossing rate given in (97) and the first-crossing rates (79) built up from two different commonly used form factors $\mathcal{R}_{\rm NG}$, the one of Matarrese et al. (2000):

$$\mathcal{R}_{\rm NG} = \exp\left[\frac{\mathcal{S}_3(\sqrt{a}\delta_c)^3}{6S}\right]\left[\sqrt{1-\frac{1}{3}(\sqrt{a}\delta_c)\mathcal{S}_3} + \frac{1}{6}\frac{(\sqrt{a}\delta_c)^2}{\sqrt{1-\frac{1}{3}(\sqrt{a}\delta_c)\mathcal{S}_3}}\frac{d\mathcal{S}_3}{d\ln\sqrt{S}}\right], \tag{99}$$

and the one of LoVerde et al. (2008):

$$\mathcal{R}_{\rm NG} = 1+\frac{1}{6}\frac{S}{\sqrt{a}\delta_c}\left[\mathcal{S}_3\left(\frac{(\sqrt{a}\delta_c)^4}{S^2} - 2\frac{(\sqrt{a}\delta_c)^2}{S} - 1\right) + \frac{d\mathcal{S}_3}{d\ln\sqrt{S}}\left(\frac{(\sqrt{a}\delta_c)^2}{S} - 1\right)\right]. \tag{100}$$

In the plots we used the conversion from the variable S to the variable M given in eq. (A2) of Neistein & Dekel (2008), while for the scale-dependence of $\mathcal{S}_3$ we used the following simple fitting formula

$$\mathcal{S}_3(S) = \frac{2.4\times 10^{-4}}{S^{0.45}} f_{\rm NL}\,, \tag{101}$$

which agrees well with LoVerde et al. (2008).

As we can see, the first-crossing rate in the case of an ellipsoidal collapse and when NG is present is not generically given by the Gaussian first-crossing rate for the ellipsoidal model multiplied by the form factor obtained from the PS approach and can differ significantly from it by $\mathcal{O}(10-50)\%$ or more at high redshift and large halo masses.

5 CONCLUSIONS

Excursion set theory provides an elegant analytical technique to describe the distribution of dark matter in our universe. When supplemented with various improvement con-

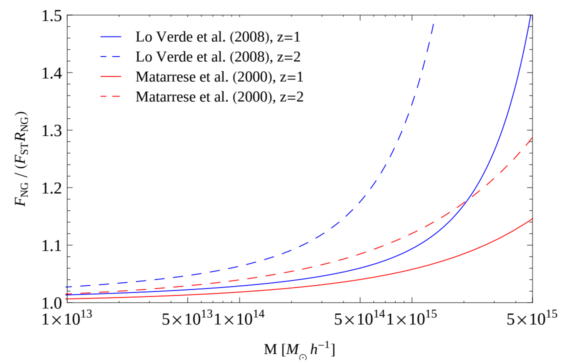

Figure 7. Ratio of the $\mathcal{F}_{\rm NG}(S)$ in (97) to the first-crossing rate given by the $\mathcal{F}_{\rm ST}(S)$ in (51) times a form factor $\mathcal{R}_{\rm NG}$, as a function of the halo mass M for $f_{\rm NL} = 100$. The form factors are those in Eq. (99) (red lines) and Eq. (100) (blue lines). We considered redshifts $z = 1$ (solid lines) and $z = 2$ (dashed lines).

cerning the physical modelisation of halo formation (such as the ellipsoidal barrier of Sheth & Tormen (1999) to take into account the triaxiality of halo collapse and the diffusing barrier of MR2 to take into account the stochasticity inherent to the process), as well as with improvements on some technical aspects (such as the inclusion of the non-Markovian dynamics introduced by the filter function), it provides a quantitative agreement with N-body simulations at the level of about 10% in most of the interesting mass range. While even more accurate results might be needed for precision cosmology, it is still remarkable that such a relatively simple theory catches quantitatively a significant part of the physics of such a complicated dynamical process as the formation of dark matter halos. The same is true if the excursion set method is applied to describe the abundances of cosmic sheets and filaments. In this paper we have extended the path integral approach proposed in MR1 for the spherical collapse case to the case of generic moving barriers using a top hat window function in wavenumber space . We have shown that, using a well controlled and systematic expansion, we can reproduce the ST halo mass function very well, therefore putting it on firmer grounds. We have also performed the computation of the first-crossing rate for

the ellipsoidal barrier in the presence of non-Gaussian initial conditions. Our result is given in eq. (97): it is fully consistent in the sense that it does not require the introduction of any form factor artificially obtained from the PS formalism based on the spherical collapse and in fact it provides a halo mass function which quantitatively differs from the one obtained from the form factor procedure.

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APPENDIX A: REPRODUCING THE FIRST-CROSSING RATE OF SHETH & TORMEN

We first compute $\Pi^{(1)}$. Using eqs. (24), (29) and (30), the expression of $\Pi_\epsilon^{(1)}(\delta_n;S_n)$ in eq. (48) can be rewritten as

$$\Pi^{(1)}_{\epsilon=0}(\delta_n;S_n) = \frac{B_n(B_n-\delta_n)}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}B_n^{(p)} \times\int_0^{S_n} dS_i\,\frac{(S_n-S_i)^{p-(3/2)}}{S_i^{3/2}} \times e^{-B_n^2/(2S_i)}e^{-(B_n-\delta_n)^2/[2(S_n-S_i)]}\,. \tag{A1}$$

Instead of computing directly this integral, we now recall that to compute the first-crossing rate (39) we need to compute the first derivative of $\Pi_\epsilon(\delta_n;S_n)$ evaluated at $\delta_n=B(S_n)$. Since the integral in eq. (A1) is finite in the limit $\delta_n\to B(S_n)$, taking the approximation $(S_n-S_i)^{p-1}\simeq(S_n)^{p-1}$ does not alter the convergence properties of the integral, but simplifies significantly its computation. This is equivalent to the approximation made by Lam & Sheth (2009), see in particular the discussion below their eq. (20). Exploiting the fact that

$$\int_0^{S_n} dS_i\,\frac{1}{S_i^{3/2}(S_n-S_i)^{1/2}} \times e^{-B^2(S_n)/(2S_i)}e^{-(B(S_n)-\delta_n)^2/(2(S_n-S_i))} = \frac{\sqrt{2\pi}}{B(S_n)}\frac{1}{S_n^{1/2}}\exp\left\{-\frac{(2B(S_n)-\delta_n)^2}{2S_n}\right\}, \tag{A2}$$

we find that $\Pi^{(1,\mathrm{ST})}_{\epsilon=0}(\delta_n;S_n)$ (where the superscript reminds us that we have approximated the integral) is given by

$$\Pi^{(1,\mathrm{ST})}_{\epsilon=0}(\delta_n;S_n) = \frac{2(B(S_n)-\delta_n)}{\sqrt{2\pi}S_n^{3/2}}e^{-(2B(S_n)-\delta_n)^2/(2S_n)} \times\sum_{p=1}^{\infty}\frac{(-S_n)^p}{p!}B_n^{(p)}\,. \tag{A3}$$

Next, we compute $\Pi^{(2)}_\epsilon(\delta_n;S_n)$. The sum over i,j in eq. (49) can be split into a sum over $i=j$ and a sum over $i<j$. The former does not contain a finite part in the continuum limit and its divergence cancels against the divergent part of the latter sum (see appendix B of MR1). Thus, we are reduced to compute the finite part of the sum over $i<j$. Proceeding as before for the calculation of $\Pi^{(1)}_\epsilon(\delta_n;S_n)$, and taking again $(S_n-S_i)^{p-1}\simeq S_n^{p-1}$ we obtain

$$\Pi^{(2,\mathrm{ST})}_{\epsilon=0}(\delta_n;S_n) = \frac{B(S_n)(B(S_n)-\delta_n)}{\pi\sqrt{2\pi}} \times\sum_{p,q=1}^{\infty}\frac{B_n^{(p)}}{p!}\frac{B_n^{(q)}}{q!}(-S_n)^{p-1}(-S_n)^{q-1} \times\int_0^{S_n} dS_i\,\frac{(S_n-S_i)e^{-\frac{B^2(S_n)}{2S_i}}}{S_i^{3/2}} \times\int_{S_i}^{S_n} dS_j\,\frac{e^{-(B(S_n)-\delta_n)^2/(2(S_n-S_j))}}{(S_j-S_i)^{3/2}(S_n-S_j)^{1/2}}\,. \tag{A4}$$

Let us indicate the integral by $\mathcal{A}(\delta_n,S_n)$. It is convenient to perform the inner integral by deriving with respect to δ_n

$$\begin{aligned}\partial_n\mathcal{A}(\delta_n,S_n) &= \int_0^{S_n} dS_i\,\frac{(S_n-S_i)e^{-\frac{B^2(S_n)}{2S_i}}}{S_i^{3/2}} \times\int_{S_i}^{S_n} dS_j\,\frac{(B(S_n)-\delta_n)e^{-\frac{(B(S_n)-\delta_n)^2}{2(S_n-S_j)}}}{(S_j-S_i)^{3/2}(S_n-S_j)^{3/2}}\\ &= \sqrt{2\pi}\int_0^{S_n} dS_i\,\frac{e^{-\frac{B^2(S_n)}{2S_i}-\frac{(B(S_n)-\delta_n)^2}{2(S_n-S_i)}}}{S_i^{3/2}(S_n-S_i)^{1/2}} \times\left[1-\frac{(B(S_n)-\delta_n)^2}{S_n-S_i}\right]\\ &= \frac{2\pi}{B(S_n)}\frac{1}{S_n^{1/2}}e^{-(2B(S_n)-\delta_n)^2/(2S_n)} \times\left[1-\frac{(2B(S_n)-\delta_n)(B(S_n)-\delta_n)}{S_n}\right],\end{aligned} \tag{A5}$$

where we used eq. (B.26) of MR1 in the second line and eqs. (A.5) of MR1 and (A2) in the third line. Integrating over δ_n we find

$$\mathcal{A}(\delta_n,S_n) = -\frac{2\pi}{S_n^{1/2}}\frac{(B(S_n)-\delta_n)}{B(S_n)}e^{-(2B(S_n)-\delta_n)^2/(2S_n)}\,, \tag{A6}$$

which can then be inserted into eq. (A4) to give

$$\Pi^{(2,\mathrm{ST})}_{\epsilon=0}(\delta_n;S_n) = -\frac{2(B(S_n)-\delta_n)^2}{\sqrt{2\pi}S_n^{5/2}} \times e^{-(2B(S_n)-\delta_n)^2/(2S_n)}\left[\sum_{p=1}^{\infty}\frac{(-S_n)^p}{p!}B_n^{(p)}\right]^2. \tag{A7}$$

A similar procedure can be used to show that higher order contributions $\Pi^{(n,\mathrm{ST})}_{\epsilon=0}$ $(n>2)$ vanish as $(B(S_n)-\delta_n)^n$ when δ_n approches the barrier value $B(S_n)$.

The calculation of the first-crossing rate is then straightforward, through eq. (39). The zero-th order contribution from $\Pi^{(0)}_{\epsilon=0}$ is given by eq. (68), while the first-order contribution from $\Pi^{(1,\mathrm{ST})}_{\epsilon=0}$ reads

$$\mathcal{F}^{(1,\mathrm{ST})}(S) = \frac{1}{\sqrt{2\pi}S^{3/2}}e^{-B^2(S)/(2S)}\sum_{p=1}^{\infty}\frac{(-S)^p}{p!}\frac{\partial^p B(S)}{\partial S^p} \tag{A8}$$

Higher-order contributions to the first-crossing rate vanish. This is already clear from the contribution arising from the second-order $\Pi^{(2,\mathrm{ST})}_{\epsilon=0}$

$$\mathcal{F}^{(2,\mathrm{ST})}(S_n) = -\left[\sum_{p=1}^{\infty}\frac{(-S_n)^p}{p!}\frac{\partial^p B_n}{\partial S_n^p}\right]^2 \frac{e^{-(2B_n-\delta_n)^2/(2S_n)}}{\sqrt{2\pi}S_n^{7/2}} \times(B_n-\delta_n)(3B_n\delta_n+2S_n-2B_n^2-\delta_n^2)\,, \quad \text{(A9)}$$

which vanishes for $\delta_n = B_n$. The total first-crossing rate for a moving barrier, in the approximation discussed above, is therefore given by

$$\mathcal{F}_{\mathrm{ST}}(S) = \frac{e^{-B^2(S)/(2S)}}{\sqrt{2\pi}S^{3/2}}\sum_{p=0}^{\infty}\frac{(-S)^p}{p!}\frac{\partial^p B(S)}{\partial S^p}\,. \quad \text{(A10)}$$

APPENDIX B: COMPUTATION OF $\Pi^{(A)}_{\epsilon=0}$, $\Pi^{(B)}_{\epsilon=0}$, $\Pi^{(C)}_{\epsilon=0}$

In this appendix we compute the contribution to $\Pi_{\epsilon=0}$ in the derivative expansion discussed in Section 3.2. The first, using the techniques discussed in MR1, is simply computed,

$$\Pi^{(a)}_{\epsilon=0}(\delta_n;S_n) = -\frac{1}{\pi}\frac{dB_n}{dS_n}B_n(B_n-\delta_n) \times\int_0^{S_n} dS_i\,\frac{1}{S_i^{3/2}(S_n-S_i)^{1/2}}\exp\left\{-\frac{B_n^2}{2S_i}-\frac{(B_n-\delta_n)^2}{2(S_n-S_i)}\right\}$$
$$= -\left(\frac{2}{\pi}\right)^{1/2}\frac{dB_n}{dS_n}\frac{(B_n-\delta_n)}{S_n^{1/2}}\exp\left\{-\frac{(2B_n-\delta_n)^2}{2S_n}\right\}. \quad \text{(B1)}$$

The second term is

$$\Pi^{(b)}_{\epsilon=0}(\delta_n;S_n) = \frac{1}{2\pi}\frac{d^2B_n}{dS_n^2}B_n(B_n-\delta_n) \times\int_0^{S_n} dS_i\,\frac{(S_n-S_i)^{1/2}}{S_i^{3/2}}\exp\left\{-\frac{B_n^2}{2S_i}-\frac{(B_n-\delta_n)^2}{2(S_n-S_i)}\right\}$$
$$= \frac{1}{2\pi}\frac{d^2B_n}{dS_n^2}(B_n-\delta_n) \times\left[\sqrt{2\pi}S_n^{1/2}e^{-(2B_n-\delta_n)^2/(2S_n)} - \pi B_n \mathrm{Erfc}\left(\frac{2B_n-\delta_n}{\sqrt{2S_n}}\right)\right], \quad \text{(B2)}$$

where the integral has been computed using eq. (109) of MR1. The last term is the most complicated. Using the α-regularization and the finite part prescription developed in Appenix. B of MR1, we find as usual that the terms in the sum with $i=j$ have a vanishing finite part, while the contribution from the terms with $i<j$ (plus an equal contribution from $i>j$) can be written as

$$\Pi^{(c)}_{\epsilon=0}(\delta_n;S_n) = \frac{B_n(B_n-\delta_n)}{\pi\sqrt{2\pi}}\left(\frac{dB_n}{dS_n}\right)^2 \times\mathcal{FP}\int_0^{S_n}dS_i\int_{S_i}^{S_n}dS_j\,\frac{(S_n-S_i)}{S_i^{3/2}(S_j-S_i)^{3/2}(S_n-S_j)^{1/2}} \times\exp\left\{-\frac{B_n^2}{2S_i}-\frac{\alpha\epsilon}{2(S_j-S_i)}-\frac{(B_n-\delta_n)^2}{2(S_n-S_j)}\right\}$$
$$= \frac{B_n(B_n-\delta_n)}{\pi\sqrt{2\pi}}\left(\frac{dB_n}{dS_n}\right)^2\int_0^{S_n}dS_i\,\frac{(S_n-S_i)}{S_i^{3/2}} \quad \text{(B3)}$$
$$\times\exp\left\{-\frac{B_n^2}{2S_i}\right\}\mathcal{FP}\int_{S_i}^{S_n}dS_j\,\frac{1}{(S_j-S_i)^{3/2}(S_n-S_j)^{1/2}} \times\exp\left\{-\frac{\alpha\epsilon}{2(S_j-S_i)}-\frac{(B_n-\delta_n)^2}{2(S_n-S_j)}\right\}, \quad \text{(B4)}$$

where $\mathcal{FP}$ denotes the finite-part prescription developed in App. B of MR1. The integral over dS_j is performed using MR1, eq. (108), and is equal to

$$\frac{\sqrt{2\pi}}{\sqrt{\alpha\epsilon}}\frac{1}{(S_n-S_i)^{1/2}}\exp\left\{-\frac{(B_n-\delta_n+\sqrt{\alpha\epsilon})^2}{2(S_n-S_i)}\right\}. \quad \text{(B5)}$$

Expanding the exponential we therefore get a singularity $1/\sqrt{\epsilon}$ (which is canceled by a similar singularity in the term of the sum with $i=j$, see MR1), and a finite part, given by

$$-\sqrt{2\pi}\frac{(B_n-\delta_n)}{(S_n-S_i)^{3/2}}e^{-(B_n-\delta_n)^2/[2(S_n-S_i)]}\,. \quad \text{(B6)}$$

The remaining integral over dS_i is performed again using MR1, eq. (108), so finally

$$\Pi^{(c)}_{\epsilon=0}(\delta_n;S_n) = -\left(\frac{2}{\pi}\right)^{1/2}(B_n-\delta_n)^2\left(\frac{dB_n}{dS_n}\right)^2\frac{1}{S_n^{1/2}} \times\exp\left\{-\frac{(2B_n-\delta_n)^2}{2S_n}\right\}. \quad \text{(B7)}$$

APPENDIX C: COMPUTATION OF $\Pi^{(1)}_{\epsilon=0}$

In this appendix we fill the missing step in the computation of $\Pi^{(1)}_{\epsilon=0}$. The issue is the computation of the integral

$$\mathcal{I}_p(a,b,S_n) \equiv \int_0^{S_n}dS_i\,S_i^{-3/2}(S_n-S_i)^{p-\frac{3}{2}} \times\exp\left\{-\frac{a^2}{2S_i}-\frac{b^2}{2(S_n-S_i)}\right\}, \quad \text{(C1)}$$

where $a\equiv B_n>0$ and $b\equiv(B_n-\delta_n)>0$. Changing the integration variable to $z=(S_n/S_i)-1$ we get

$$\mathcal{I}_p(a,b,S_n) = S_n^{p-2}\exp\left\{-\frac{a^2+b^2}{2S_n}\right\}\int_0^{\infty}dz \times\left(\frac{1}{z^{3/2}}+\frac{1}{z^{1/2}}\right)\left(\frac{z}{1+z}\right)^p \times\exp\left\{-\left(\frac{a^2}{2S_n}\right)z-\left(\frac{b^2}{2S_n}\right)\frac{1}{z}\right\}. \quad \text{(C2)}$$

For $p=0,1$ the integral can be performed exactly (see eq. 9.471.12 of Gradstein & Ryzhik (1980)) and we get[3]

$$\mathcal{I}_0(a,b,S_n) = \frac{(2\pi)^{1/2}}{S_n^{3/2}}\frac{a+b}{ab}e^{-(a+b)^2/(2S_n)}\,, \quad \text{(C3)}$$

$$\mathcal{I}_1(a,b,S_n) = \frac{(2\pi)^{1/2}}{S_n^{1/2}}\frac{1}{a}e^{-(a+b)^2/(2S_n)}\,. \quad \text{(C4)}$$

For $p\geqslant 2$ we have not been able to compute the integral exactly. However, as discussed in the text, for computing the first-crossing rate it is sufficient to evaluate it at $b=0$. The resulting integral can be computed (e.g. using Mathematica)

[3] These integrals were already computed exactly in a different way in MR1. We thank Ruth Durrer for suggesting this more direct derivation.

in terms of the confluent hypergeometric function $U(a,b,z)$,

$$\mathcal{I}_p(a,0,S_n) = S_n^{p-2}\,\frac{\sqrt{2S_n}}{a}\,e^{-a^2/(2S_n)} \times\Gamma\left(p-\frac{1}{2}\right)U\left(p-1,\frac{1}{2},\frac{a^2}{2S_n}\right). \quad \text{(C5)}$$

Observe that $U(0,b,z)=1$ and $\Gamma(1/2)=\sqrt{\pi}$, so eq. (C5) also reproduces correctly $\mathcal{I}_p(a,0,S_n)$ when $p=1$. It is also useful the limit

$$\mathcal{I}_p(0,0,S_n) \equiv \mathcal{FP}\lim_{a\to 0}\mathcal{I}_p(a,0,S_n) = -\pi c_p S_n^{p-2}\,, \quad \text{(C6)}$$

where the coefficients c_p are given by

$$c_p = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(p-\frac{1}{2}\right)}{\Gamma\left(p-1\right)}\,. \quad \text{(C7)}$$

APPENDIX D: COMPUTATION OF THE GENERAL TERM $\Pi^{(M)}_{\epsilon=0}$ IN THE LIMIT $(B_N-\delta_N)\to 0$

The general term $\Pi^{(m)}_{\epsilon=0}$ is given by

$$\Pi^{(m)}_{\epsilon=0} = \frac{1}{m!}\sum_{p_1,\dots,p_m=1}^{\infty}\frac{B_n^{(p_1)}\cdots B_n^{(p_m)}}{p_1!\cdots p_m!} \times\sum_{i_1,\dots,i_m=1}^{n-1}(S_{i_1}-S_n)^{p_1}\cdots(S_{i_m}-S_n)^{p_m} \times\int_{-\infty}^{B_n}d\delta_1\cdots d\delta_{n-1}\partial_{i_1}\cdots\partial_{i_m}W^{\rm gm}\,. \quad \text{(D1)}$$

The last integral is equal to

$$\int_{-\infty}^{B_n}d\delta_1\cdots d\delta_{n-1}\partial_{i_1}\cdots\partial_{i_m}W^{\rm gm} = \Pi^{\rm gm}(\delta_0,B_n,S_{i_1})\Pi^{\rm gm}(B_n,B_n,S_{i_2}-S_{i_1})\cdots \cdots\Pi^{\rm gm}(B_n,B_n,S_{i_m}-S_{i_{m-1}})\Pi^{\rm gm}(B_n,\delta_n,S_n-S_{i_m}).$$

Using eqs. (29)-(31) for $\Pi^{\rm gm}$, eq. (D1) becomes

$$\Pi^{(m)}_{\epsilon=0} = \frac{1}{m!}\frac{B_n(B_n-\delta_n)}{2^{\frac{m-1}{2}}\pi^{\frac{m+1}{2}}}\sum_{p_1,\dots,p_m=1}^{\infty}(-1)^{p_1+\dots+p_m} \times\frac{B_n^{(p_1)}\cdots B_n^{(p_m)}}{p_1!\cdots p_m!}\mathcal{J}^{(m)}_{p_1,\dots,p_m}(B_n,S_n) +\mathcal{O}(B_n-\delta_n)^2\,, \quad \text{(D2)}$$

where

$$\mathcal{J}^{(m)}_{p_1,\dots,p_m}(B_n,S_n)\equiv\mathcal{FP}\int_0^{S_n}dS_{i_1}\frac{(S_n-S_{i_1})^{p_1}}{S_{i_1}^{3/2}}e^{-\frac{B_n^2}{2S_{i_1}}} \times\int_{S_{i_1}}^{S_n}dS_{i_2}\frac{(S_n-S_{i_2})^{p_2}}{(S_{i_2}-S_{i_1})^{3/2}}\times(\dots) \times\int_{S_{i_{m-1}}}^{S_n}dS_{i_m}\frac{(S_n-S_{i_m})^{p_m-3/2}}{(S_{i_m}-S_{i_{m-1}})^{3/2}}\,. \quad \text{(D3)}$$

We have only considered the finite parts from the sum with $i_1<i_2<\dots<i_m$, because the divergent parts all cancel. A priori, we cannot exclude that there may be other finite contributions to the sum coming from terms with $i_1<\dots<i_k=i_{k+1}<\cdots i_m$. However, we expect the contribution we compute here as representative of the correct result.

The integral $\mathcal{J}^{(m)}_{p_1,\dots,p_m}(B_n,S_n)$ satisfies the recursion relation

$$\mathcal{J}^{(m)}_{p_1,\dots,p_m}(B_n,S_n) = \int_0^{S_n}dS_i\frac{e^{-B_n^2/(2S_i)}(S_n-S_i)^{p_1}}{S_i^{3/2}} \times\mathcal{J}^{(m-1)}_{p_2,\dots,p_m}(0,S_n-S_i)\,. \quad \text{(D4)}$$

Let us set

$$\mathcal{J}^{(m)}_{p_1,\dots,p_m}(0,y) = (-\pi)^m c_{p_1,\dots,p_m}\,y^{p_1+\dots+p_m-\frac{m+3}{2}}\,, \quad \text{(D5)}$$

where the coefficients c are now to be determined. We insert the ansatz above into the recursion relation (D4) for $\mathcal{J}^{(m+1)}_{p_1,\dots,p_m}(B_n,S_n)$ and obtain

$$\mathcal{J}^{(m+1)}_{p_1,\dots,p_{m+1}}(B_n,S_n) = (-\pi)^m c_{p_2,\dots,p_{m+1}} \times\int_0^{S_n}dS_i(S_n-S_i)^{p_1+\dots+p_{m+1}-\frac{m+3}{2}}S_i^{-3/2}e^{-B_n^2/(2S_i)} \quad \text{(D6)}$$

The previous integral is solved with the substitution $z=(S_n/S_i)-1$ and it evaluates to

$$S_n^{\sum_{k=1}^{m+1}p_k-\frac{m}{2}-2}\int_0^\infty dz\frac{z^{\sum_{k=1}^{m+1}p_k-\frac{m+3}{2}}}{(1+z)^{\sum_{k=1}^{m+1}p_k-\frac{m}{2}-1}}e^{-\frac{B_n^2}{2S_n}(1+z)} = S_n^{\sum_{k=1}^{m+1}p_k-\frac{m}{2}-2}e^{-\frac{B_n^2}{2S_n}}\sqrt{\frac{2S_n}{B_n^2}}\,\Gamma\left(\sum_{k=1}^{m+1}p_k-\frac{m+1}{2}\right) \times U\left(\sum_{k=1}^{m+1}p_k-\frac{m+1}{2},\frac{1}{2},\frac{B_n^2}{2S_n}\right), \quad \text{(D7)}$$

therefore eq. (D6) becomes

$$\mathcal{J}^{(m+1)}_{p_1,\dots,p_{m+1}}(B_n,S_n) = (-\pi)^m c_{p_2,\dots,p_{m+1}}S_n^{\sum_{k=1}^{m+1}p_k-\frac{m}{2}-2} \times\sqrt{\frac{2S_n}{B_n^2}}e^{-\frac{B_n^2}{2S_n}}\Gamma\left(\sum_{k=1}^{m+1}p_k-\frac{m+1}{2}\right) \times U\left(\sum_{k=1}^{m+1}p_k-\frac{m+1}{2},\frac{1}{2},\frac{B_n^2}{2S_n}\right). \quad \text{(D8)}$$

We can evaulate eq. (D8) in the limit $B_n^2/(2S_n)\to 0$, and retain the finite part only (as the divergent terms all cancel in the end):

$$\mathcal{J}^{(m+1)}_{p_1,\dots,p_{m+1}}(0,y) = -2\sqrt{\pi}(-\pi)^m c_{p_2,\dots,p_{m+1}} \times\frac{\Gamma\left(\sum_{k=1}^{m+1}p_k-\frac{m}{2}-\frac{1}{2}\right)}{\Gamma\left(\sum_{k=1}^{m+1}p_k-\frac{m}{2}-1\right)}y^{\sum_{k=1}^{m+1}p_k-\frac{m}{2}-2}\,. \quad \text{(D9)}$$

On the other hand, the left-hand side of the previous relation can be expressed by (D5) and we then arrive at a recursion relation for the coefficients c (after relabelling $m\to m-1$ for convenience):

$$c_{p_1,\dots,p_m} = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\sum_{k=1}^m p_k-\frac{m}{2}\right)}{\Gamma\left(\sum_{k=1}^m p_k-\frac{m+1}{2}\right)}c_{p_2,\dots,p_m}\,, \quad \text{(D10)}$$

which is valid for $m\geqslant 2$, while for $m=1$ we have already found in (C7)

$$c_p = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(p-\frac{1}{2}\right)}{\Gamma\left(p-1\right)}\,. \quad \text{(D11)}$$

Equations (D10)-(D11) define recursively the coefficients c and it is possible to find them easily up to any desired order. As the c appear in the generic integral (D8), which in turn

appears in (D2), it is then possible to write down the result for the generic term $\Pi^{(m)}$:

$$\begin{aligned}
\Pi^{(m)}_{\epsilon=0} &= \frac{(B_n-\delta_n)e^{-\frac{B_n^2}{2S_n}}}{m!\,2^{\frac{m}{2}-1}\pi^{\frac{3-m}{2}}}\sum_{p_1,\dots,p_m=1}^{\infty}(-1)^{\sum_{k=1}^m p_k+m+1} \\
&\times \frac{B_n^{(p_1)}\cdots B_n^{(p_m)}}{p_1!\cdots p_m!}c_{p_2,\dots,p_m}S_n^{\sum_{k=1}^m p_k-\frac{m}{2}-1} \\
&\times \Gamma\left(\sum_{k=1}^m p_k-\frac{m}{2}\right)U\left(\sum_{k=1}^m p_k-\frac{m+1}{2},\frac{1}{2},\frac{B_n^2}{2S_n}\right) \\
&+ \mathcal{O}(B_n-\delta_n)^2. \qquad \text{(D12)}
\end{aligned}$$

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